Extended Boltzmann Equations for Light Particles Reacting with a Medium

M. Groppi* and A. Rossani†

* Dipartimento di Matematica, Univ. Parma - V. D’Azeglio, 85 - 43100 Parma - ITALY
† Istituto Nazionale di Fisica della Materia - Politecnico Torino
Corso Duca degli Abruzzi 24 - 10129 Torino - ITALY

Abstract. We consider two species of light test particles interacting with a medium constituted by two species of much heavier field particles. A reversible bimolecular chemical reaction is assumed to occur, with conservation of mass, momentum and total energy. In the Lorentz gas limit, the resulting model is described by a system of two extended linear Boltzmann equations for the light test particles. The main properties of the collision operators are discussed and collision invariants are properly introduced. Equilibria are determined and turn out to be defined up to an arbitrary isotropic shape function. Stability and uniqueness of the equilibrium associated to any given initial datum are investigated in the space homogeneous case.

INTRODUCTION

In the last years, the extended linear Boltzmann equation has received some interest from researchers in the field of transport theory. This equation was derived [1] starting from the nonlinear system of kinetic equations for a mixture of particles A (mass \(m\)) and B (mass \(M\)), where B is endowed with two internal energy levels: fundamental and excited. The problem is nonconservative since both elastic and inelastic scattering are taken into account, and becomes linear once the particles B are considered as a fixed background, so that their distribution functions are assumed to be known. Moreover, if we consider \(M \gg m\), it is possible to let \(M \to \infty\). The resulting model is such that test particles A can gain or lose a fixed amount of kinetic energy by interacting with field particles B which, at the same time, undergo a transition between the two internal quantum states. In [2] it was pointed out that the collisional integral is mathematically equivalent to the one describing electrons which emit or absorb monoenergetic phonons in a crystal lattice.

On the other hand, the kinetic theory of chemically reacting gases has been widely investigated starting from the Sixties. Just to recall some of the main contributions to this field, we may quote the pioneering paper by Ross and Mazur [3] as the first application of the Chapman-Enskog expansion in this context, followed by a series of papers by Shizgal and Karplus [4], with a more refined treatment of that expansion. The Grad’s method for a bimolecular chemical reaction is due to Moreau [5], and, moreover, a series of papers by Xystris and Dahler [6] deal with the Enskog equations for reacting gases. With regard to equilibria and their stability, the first weak result concerning the H-theorem was given by Polak and Khachoyan [7]. More recently, Rossani and Spiga [8] have given a clear characterization of equilibrium distributions and of the underlying H-theorem; the extension to many-level molecules has been performed by Groppi and Spiga [9].

Combining these two lines of research, in the present paper we investigate the following situation. Light particles A (mass \(m_1\)) and C (mass \(m_3\)) interact with a medium constituted by much heavier particles B (mass \(M_2\)) and D (mass \(M_4\)). Particles A, B, C, D will be labeled in the sequel by the indices 1, 2, 3, 4. The reaction

\[ A + B \rightleftharpoons C + D \]  

(1)

is assumed to occur, with conservation of mass, momentum and total energy. Each of these particles is endowed with an internal energy of chemical link \(E_i\), \(i = 1, \cdots, 4\). As an example of such a situation, we could quote
the reversible bimolecular reaction \( H_2 + Cl \leftrightarrow H + HCl \) in a background formed by \( Cl \) and \( HCl \), which are much heavier than \( H_2 \) and \( H \).

We start from the nonlinear system of kinetic equations for the mixture of particles \( A, B, C, D \), and, consistently with the above assumption, we let \( M_2 \to \infty \) and \( M_4 \to \infty \), keeping \( (M_2/M_4) \to 1 \) (Lorentz gas limit). Let us call \( f_2(v) \) and \( f_4(v) \) the distribution functions of particles 2 and 4, respectively. Because of the further assumption that particles 2 and 4 are much more numerous and can be treated as a background, it is reasonable to assume that their distribution functions are equilibrium Maxwellians with no drift, so that, when we let the masses tend to infinity, we get Dirac’s distributions

\[
f_2(v) = N_2 \delta(v), \quad f_4(v) = N_4 \delta(v),
\]

where \( N_2 \) and \( N_4 \) are the number densities (given constants) of particles 2 and 4, respectively. The resulting model is described now by a system of two linear Boltzmann equations for test particles 1 and 3.

This paper is organized as follows. In the second section we deduce the system of extended linear Boltzmann equations, in the Lorentz gas limit, starting from the full nonlinear description [8]. In the following section we introduce collision invariants and derive the relevant macroscopic conservation equations. In the final section, we investigate equilibria and their stability by means of suitable Lyapunov functionals.

With respect to the nonlinear kinetic description, some of its main features are reproduced, while the Lorentz limit introduces new interesting aspects, such as no correlation between equal particles at different speeds, implying non uniqueness of equilibria, defined up to an arbitrary isotropic function.

**DERIVATION OF THE EXTENDED LINEAR BOLTZMANN EQUATIONS**

The nonlinear integrodifferential Boltzmann equations governing the evolution of the distribution function \( f = (f_1, f_2, f_3, f_4) \) for a mixture of 4 chemically reacting gases read as [8]:

\[
\frac{\partial f_i}{\partial t} + v \cdot \frac{\partial f_i}{\partial x} = \tilde{J}_i[f] + \tilde{Q}_i[f] \quad i = 1, \ldots, 4.
\]

On the right hand side of (3), \( \tilde{J}_i[f] \), \( i = 1, \ldots, 4 \), stands for the chemical collision integral, and \( \tilde{Q}_i[f] \), \( i = 1, \ldots, 4 \), is the elastic contribution.

Again from ref. [8], for \( i = 1 \) we have

\[
\tilde{J}_1[f](v) = \int \int_{R^3 \times S^2} U \left( g^2 - \frac{2\Delta E}{\mu_{12}} \right) \nu_{12}^{34}(g, n \cdot n') \left[ \left( \frac{\mu_{12}}{\mu_{34}} \right)^3 f_2(v_{12}^{34}) f_4(w_{12}^{34}) - f_1(v) f_2(w) \right] dv dn'^{1/2}
\]

where \( \Delta E = E_3 + E_4 - E_1 - E_2 \) (and it can always be assumed that \( \Delta E \geq 0 \)), \( \mu_{ij} = \frac{m_im_j}{m_i + m_j} \) is the reduced mass, and \( \nu_{12}^{34}(g, n \cdot n') \) is the differential collision frequency relevant to the direct (endothermic) reaction. The microreversibility condition

\[
\mu_{12}^2 g_{12}^{34}(g, n \cdot n') = \mu_{34}^2 g_{12}^{34} U \left( g^2 - \frac{2\Delta E}{\mu_{12}} \right) \frac{1}{\nu_{34}^{12}(g_{12}^{34}, n \cdot n')}
\]

eliminates one of the collision frequencies; here \( g_{12}^{34} = \left[ \frac{\mu_{12}}{\mu_{34}} \left( g^2 - \frac{2\Delta E}{\mu_{12}} \right) \right]^{1/2} \) is the post-collisional relative speed in the direct process, while \( n \) and \( n' \) are the unit vectors of the relative velocities before and after collision, respectively. The velocities after collision are given by

\[
v_{12}^{34} = \frac{m_1}{m_1 + m_2} v + \frac{m_2}{m_1 + m_2} w + \frac{m_4}{m_3 + m_4} g_{12}^{34} n'
\]
\[
w_{12}^{34} = \frac{m_1}{m_1 + m_2} v + \frac{m_2}{m_1 + m_2} w - \frac{m_3}{m_3 + m_4} g_{12}^{34} n'.
\]

The unit step function \( U \) accounts for the threshold in the endothermic reaction, that is prevented in case of insufficient impinging kinetic energy of the relative motion. The other chemical contributions \( \tilde{J}_i \) relevant to \( i = 2, 3, 4 \) can be obtained from \( \tilde{J}_1 \) by proper permutation of indices [8].
The nonlinear elastic collision integrals in (3) for \( i = 1 \) is given by
\[
\hat{Q}_1[f](v) = \sum_{\ell=1}^{4} \int_{\mathbb{R}^3 \times S^2} \nu_{\ell}^{(v, n \cdot n')} \left[ f_{\ell} (v_{\ell}^{(v, n \cdot n')}) - f_{\ell} (v) \right] \, dw \, dn',
\] (7)
where the expressions for post-collisional velocities \( v_{\ell}^{(v, n \cdot n')} \) and \( w_{\ell}^{(v, n \cdot n')} \) are well known [10]; contributions \( \hat{Q}_i \) for \( i = 2, 3, 4 \) can be obtained analogously.

It has been proved [8] that the equilibrium distributions \( f = (f_1, f_2, f_3, f_4) \) for the set of nonlinear equations (3) are Maxwellsians, in which the number densities satisfy the following condition
\[
\frac{n_3 \, N_4}{n_1 \, N_2} = \left( \frac{m_3 \, M_4}{m_1 \, M_2} \right)^{3/2} \exp \left( -\frac{\Delta E}{KT} \right),
\] (8)
representing the mass action law for the considered reaction.

Now observe that, in the Lorentz gas limit for the considered problem (that is \( M_2 \to \infty \) and \( M_4 \to \infty \) with \( M_2/M_4 \to 1 \) we have
\[
\begin{align*}
\mu_{12} &\to m_1, & \mu_{34} &\to m_3, & g_n &\to v = v \Omega, \\
g_{12}^{34} &\to v_{13} = \left[ \frac{m_1}{m_3} \left( v^2 - \frac{2 \Delta E}{m_1} \right) \right]^{1/2}, & g_{34}^{12} &\to v_{31}^{+} = \left[ \frac{m_3}{m_1} \left( v^2 + \frac{2 \Delta E}{m_3} \right) \right]^{1/2},
\end{align*}
\] (9)
By inserting the limiting expression (2) into (4) and (7) and by taking into account the implication (9) of the Lorentz gas assumption, we obtain a system of two linear Boltzmann equations for the evolution of the distribution functions of test particles
\[
\frac{\partial f_i}{\partial t} + v \cdot \frac{\partial f_i}{\partial x} = J_i[f_1, f_3] + Q_i[f_i] \quad i = 1, 3
\] (10)
or, explicitly,
\[
\begin{align*}
\frac{\partial f_1}{\partial t} + v \cdot \frac{\partial f_1}{\partial x} &= \int_{S^2} \{ K_1[f_1, f_3](v, \Omega') + R_1[f_1](v, \Omega') \} \, d\Omega' \\
\frac{\partial f_3}{\partial t} + v \cdot \frac{\partial f_3}{\partial x} &= \int_{S^2} \{ K_3[f_1, f_3](v, \Omega') + R_3[f_3](v, \Omega') \} \, d\Omega'
\end{align*}
\] (11)
where the chemical reaction and elastic collision kernels are respectively given by
\[
K_1[f_1, f_3] = U \left( v^2 - \frac{2 \Delta E}{m_1} \right) \nu_{12}^3(v, \Omega \cdot \Omega') \left[ \frac{m_1}{m_3} \right]^3 N_4 f_3(v_{13}^\Omega - N_2 f_1(v))
\] (12)
\[
R_1[f_1] = [N_2 \nu_{12}^{13}(v, \Omega \cdot \Omega') + N_4 \nu_{14}^{14}(v, \Omega \cdot \Omega') ] [f_1(v \Omega') - f_1(v)]
\] (13)
and, by using the microreversibility condition,
\[
K_3[f_1, f_3] = \left( \frac{m_1}{m_3} \right)^2 \nu_{12}^3(v_{13}^\Omega, \Omega \cdot \Omega') \left[ \frac{m_3}{m_1} \right]^3 N_2 f_1(v_{34}^\Omega - N_4 f_3(v))
\] (14)
\[
R_3[f_3] = [N_2 \nu_{34}^{13}(v, \Omega \cdot \Omega') + N_4 \nu_{34}^{14}(v, \Omega \cdot \Omega') ] [f_3(v \Omega') - f_3(v)]
\] (15)

**COLLISION INVARIANTS AND MACROSCOPIC EQUATIONS**

Having in mind concepts and ideas from kinetic theory [10], let us introduce an arbitrary pair of smooth functions \( \phi(v) = (\phi_1(v), \phi_3(v)) \) and consider the functional \( C(\phi) = C^{ch}(\phi) + C^{ef}(\phi) \), where
Based on these expressions, we can define as collision invariant any pair \((\phi_1(v), \phi_3(v))\) satisfying the requirements

\[
\begin{align*}
\phi_1(v) &= \phi_1(v) \\
\phi_3(v) &= \phi_3(v) \\
\phi_3(v) &= \phi_1(v) \\
\forall v \in \mathbb{R}^+.
\end{align*}
\]

(18)

They must be isotropic, and are defined up to an arbitrary smooth function of \(v\). In this way, of course, \(C[\phi] = 0\) whenever \(\phi\) is a collision invariant.

In particular, if we multiply the kinetic equations (11) by the collision invariant \(\phi_1(v) = \phi_3(v) = 1\), sum the two equations, and integrate over the kinetic variable \(v\), we find, in space homogeneous conditions,

\[
\dot{n}_1 + \dot{n}_3 = 0
\]

(19)

that is, the total number of test particles is conserved. Moreover, starting from the collision invariant \(\phi_1(v) = \frac{1}{2} m_1 v^2, \phi_3(v) = \frac{1}{2} m_1 (v_3^1)^2 = \frac{1}{2} m_3 v^2 + \Delta E\), we obtain

\[
\dot{E} = -\dot{n}_3 \Delta E = \dot{n}_1 \Delta E
\]

(20)

that represents the balance equation for overall test particle kinetic energy

\[
\mathcal{E} = \int_{\mathbb{R}^3} \left[ \frac{1}{2} m_1 v^2 f_1(v) + \frac{1}{2} m_3 v^2 f_3(v) \right] dv.
\]

Let us now consider \(\phi_1(v) = \log [N_2 f_1(v)/m_1^2], \phi_3(v) = \log [N_4 f_3(v)/m_3^2]\) and define the functional

\[
W[f_1, f_3] = C(\phi_1, \phi_3)
\]

\[
= \int_{\mathbb{R}^3} K_1[f_1, f_3] \log \left[ \frac{f_1(v)}{f_3(v_1^3 \Omega')} \right] \frac{N_2}{N_4} \left( \frac{m_3}{m_1} \right)^3 dv \Omega' \\
+ \frac{1}{2} \int_{\mathbb{R}^3} \left\{ R_1[f_1] \log \frac{f_1(v)}{f_3(v_1 \Omega')} + R_3[f_3] \log \frac{f_3(v)}{f_3(v_1 \Omega')} \right\} dv \Omega'.
\]

(21)

After some algebra, we obtain, under standard positivity assumptions on collision frequencies and distribution functions

\[
W = -\int \int U \left( v^2 - \frac{2 \Delta E}{m_1} \right) \nu_{12}^{[3]}(v, \Omega \cdot \Omega') N_4 \left( \frac{m_1}{m_3} \right)^3 f_3(v_1^3 \Omega') \\
\cdot \left[ \frac{f_1(v)}{f_3(v_1^3 \Omega')} \right] \frac{N_2}{N_4} \left( \frac{m_3}{m_1} \right)^3 - 1 \log \left[ \frac{f_1(v)}{f_3(v_1^3 \Omega')} \right] \frac{N_2}{N_4} \left( \frac{m_3}{m_1} \right)^3 dv \Omega' \\
- \frac{1}{2} \int \int \left[ N_2 \nu_{12}^{[3]}(v, \Omega \cdot \Omega') + N_4 \nu_{14}^{[3]}(v, \Omega \cdot \Omega') \right] f_1(v \Omega') \left[ \frac{f_1(v)}{f_1(v \Omega')} - 1 \right] \log \frac{f_1(v)}{f_1(v \Omega')} dv \Omega' \\
- \frac{1}{2} \int \int \left[ N_2 \nu_{22}^{[3]}(v, \Omega \cdot \Omega') + N_4 \nu_{24}^{[3]}(v, \Omega \cdot \Omega') \right] f_3(v \Omega') \left[ \frac{f_3(v)}{f_3(v \Omega')} - 1 \right] \log \frac{f_3(v)}{f_3(v \Omega')} dv \Omega'.
\]

(22)
Since, for \( x > 0 \), the convex function \((x - 1)\log x\) is non-negative, and vanishes only at \( x = 1 \), from (22) we can establish the sign of \( W \) and generalize the classical Boltzmann inequality.

**Theorem 1** The functional \( W[f_1, f_3] \) defined by (21) is such that \( W \leq 0 \) \( \forall (f_1, f_3) \), and more precisely \( W = 0 \) if and only if

\[
\begin{align*}
f_1(v) &= f_1(v) \\
f_3(v) &= f_3(v) \\
\frac{N_1 f_3(v)}{m_3^2} &= \frac{N_2 f_1(v_3^+)}{m_1^3} \quad \forall v \in \mathbb{R}^+.
\end{align*}
\]  

(23)

Notice that chemical reaction establishes a correlation between particles 1 at a speed \( v \geq \sqrt{2\Delta E/m_1} \) and particles 3 at the corresponding speed \( w = v_1^+ \geq 0 \). Particles 1 at a speed \( v < \sqrt{2\Delta E/m_1} \) are not involved in the correlation since chemical reaction is not allowed to them.

A further application of the collision rate expressions (16) and (17) is the derivation of the macroscopic conservation equation that is in order in our case. Consider an arbitrary collision invariant \((\phi_1, \phi_3)\), that can be written, without loss of generality, as \( \phi_1(v) = \Phi(v) \), \( \phi_3(v) = \Phi(v_{31}^+) \), where \( \Phi \) is an arbitrary isotropic function. Then the collision rate equation \( C[\phi] = 0 \) can be rewritten as

\[
\int_0^{+\infty} \Phi(v) v^2 \left[ \int \left\{ [J_1[f_1, f_3]](v\Omega) + Q_1[f_1](v\Omega) \right\} \\
+ \frac{m_1}{m_3} U \left( v^2 - \frac{2\Delta E}{m_1} \right) \frac{v_1^+}{v} \left\{ [J_3[f_1, f_3]](v_1^+\Omega) + Q_3[f_3](v_1^+\Omega) \right\} \right] d\Omega \right] dv = 0.
\]  

(24)

By taking into account the arbitrariness of \( \Phi(v) \) and the kinetic equations (11) we finally get the following continuity equation

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left[ \int \{ v\Omega f_1(v\Omega) + \frac{m_1}{m_3} U \left( v^2 - \frac{2\Delta E}{m_1} \right) \frac{v_1^+}{v} f_3(v_1^+\Omega) \} d\Omega \right] = 0,
\]  

(25)

where \( \rho(v) \) is a partial density at speed \( v \) collecting in appropriate way the two species of test particles and is defined as

\[
\rho(v) = \int \left\{ f_1(v\Omega) + \frac{m_1}{m_3} U \left( v^2 - \frac{2\Delta E}{m_1} \right) \frac{v_1^+}{v} f_3(v_1^+\Omega) \right\} d\Omega.
\]  

(26)

**EQUILIBRIUM AND STABILITY**

A collision equilibrium can be defined, like in [10], as any pair of distribution functions \((f_1, f_3)\) for which

\[
\begin{align*}
J_1[f_1, f_3] + Q_1[f_1] &= 0 \\
J_3[f_1, f_3] + Q_3[f_3] &= 0
\end{align*}
\]

\( \forall v \in \mathbb{R}^3 \).  

(27)

By means of the Boltzmann inequality, the following theorem can be proved:

**Theorem 2** The pair \((f_1, f_3)\) is a collision equilibrium if and only if satisfies conditions (23), that is equilibria are isotropic functions defined up to an arbitrary shape function \( F(v) \) (such that \( 4\pi \int F(v)v^2dv = 1 \)) as follows

\[
\begin{align*}
f_1(v) &= n_1 F(v) \\
f_3(v) &= \left( \frac{m_3^3}{m_1^3} \right) \frac{N_2}{N_4} n_1 F(v_{31}^+).
\end{align*}
\]  

(28)
This theorem represents the extended version of the detailed balance principle that is valid in the present context, binding collision invariants to equilibria.

In particular, by choosing \( F(v) = \left( \frac{m_1}{2\pi KT} \right)^{3/2} \exp \left( -\frac{m_1}{2KT} v^2 \right) \), in such a way that the equilibrium distribution \( f_1(v) = n_1 F(v) \) is a Maxwellian at temperature \( T \), correspondingly also \( f_3 \) turns out to be the Maxwellian at the same temperature

\[
f_3(v) = n_3 \left( \frac{m_3}{2\pi KT} \right)^{3/2} \exp \left( -\frac{m_3}{2KT} v^2 \right),
\]

where the number density \( n_3 \) is related to \( n_1 \) by

\[
n_3 = n_1 \frac{N_2}{N_4} \left( \frac{m_3}{m_1} \right)^{3/2} \exp \left( -\frac{\Delta E}{KT} \right).
\]

It is interesting to observe that the present relationship (30) could also be obtained from the nonlinear mass action law (8) in the Lorentz gas limit.

In general, for an arbitrary equilibrium characterized by \( F(v) \), the following relationship between number densities is in order

\[
n_3 = n_1 \frac{N_2}{N_4} \left( \frac{m_3}{m_1} \right)^2 4\pi \int_0^{+\infty} U(v^2 - 2\Delta E/m_1) \frac{v_3}{v} F(v) v^2 \, dv
\]

which can be viewed as the general mass action law holding in this linear limiting context.

We restrict ourselves now to the space homogeneous evolution problem starting from a given initial state. The continuity equation (25) yields the first integral \( \rho(v) = \rho^0(v) \), where \( \rho^0 \) is uniquely determined from (26) when initial conditions \((f_1^0, f_3^0)\) are assigned. At equilibrium, from (28), we obtain

\[
\rho^\text{eq}(v) = 4\pi n_1 F(v) \left[ 1 + U \left( v^2 - \frac{2\Delta E}{m_1} \right) \left( \frac{m_3}{m_1} \right)^2 \frac{v_3}{v} \frac{N_2}{N_4} \right].
\]

Then, by equating (31) to \( \rho^0(v) \) we have

**Theorem 3** There exists a unique equilibrium associated to any initial datum \((f_1^0, f_3^0)\), given by

\[
\begin{align*}
  f_1^{eq}(v) &= \frac{\rho^0(v)}{4\pi} \left[ 1 + U \left( v^2 - \frac{2\Delta E}{m_1} \right) \left( \frac{m_3}{m_1} \right)^2 \frac{v_3}{v} \frac{N_2}{N_4} \right]^{-1} \\
  f_3^{eq}(v) &= \left( \frac{m_3}{m_1} \right)^3 \frac{N_2}{N_4} f_1^{eq}(v^{1/3}).
\end{align*}
\]

In order to study stability under collision of the equilibrium \((f_1^{eq}, f_3^{eq})\) associated to a given initial condition, introduce the \( H \)-functional

\[
H[f_1, f_3] = \int \left\{ f_1(v) \log \frac{f_1(v)N_2}{m_1^3} + f_3(v) \log \frac{f_3(v)N_4}{m_3^3} \right\} dv.
\]

It is possible to prove that the following H-Theorem holds.

**Theorem 4** The functional \( H \) in (33) is a strict Lyapunov functional in space homogeneous conditions for the unique equilibrium \((f_1^{eq}, f_3^{eq})\) determined by \((f_1^0, f_3^0)\), in the sense that

\[
\dot{H}[f_1, f_3] \leq 0, \quad \text{and} \quad \dot{H}[f_1, f_3] = 0 \quad \text{iff} \quad (f_1, f_3) = (f_1^{eq}, f_3^{eq})
\]

\[
H[f_1, f_3] - H[f_1^{eq}, f_3^{eq}] > 0 \quad \text{for any} \quad (f_1, f_3) \neq (f_1^{eq}, f_3^{eq}).
\]
The proof follows similar steps as in [8]. First, we can see by direct computation that $\dot{H} = W$, and then the former requirement in (34) follows immediately from Theorem 1. For the latter, if we write $H$ in the form $H = \int \mathcal{H} \, dv$, bearing in mind the conservation equation, we obtain after some algebra:

$$\int \left\{ \left( \frac{\partial \mathcal{H}}{\partial f_1} \right)^{eq} [f_1(v) - f_1^{eq}(v)] + \left( \frac{\partial \mathcal{H}}{\partial f_3} \right)^{eq} [f_3(v) - f_3^{eq}(v)] \right\} \, dv = 0$$

and then it is possible to write

$$H[f_1, f_3] - H[f_1^{eq}, f_3^{eq}] = \int (\mathcal{H} - \mathcal{H}^L) \, dv$$

where $\mathcal{H}^L$ denote the first-order Taylor polynomial of $\mathcal{H}$. Since $\mathcal{H}$ is a convex function for $f_1 > 0, f_3 > 0$, the integrand in (36) is non-negative, and vanishes if and only if $(f_1, f_3)$ coincides with $(f_1^{eq}, f_3^{eq})$, which completes the proof.

We can observe that the property $\dot{H} \leq 0$ may be linked to the entropy law for the present non-isolated system of test particles.

It is possible to prove that also

$$H^*[f_1, f_3] = \frac{1}{2} \int \left\{ [f_1(v)]^2 \frac{N_4}{m_1^3} + [f_3(v)]^2 \frac{N_4}{m_3^3} \right\} \, dv$$

is a strict Lyapunov functional for the present problem.

**CONCLUSIONS**

A linear Boltzmann model can be constructed in the case of light test particles reacting with a medium of heavy field particles, starting from the nonlinear kinetic description of reversible bimolecular reactions in chemically reacting gases. The Lorentz limit introduces, especially from a mathematical point of view, new interesting aspects typical of linear transport theory. Some of the classical problems of neutron transport theory can be revisited in such a new context. Indeed, this is a kind of two-species monoenergetic transport [11]. On the other hand, by dropping the strong assumption that field particles are described by Dirac’s delta distribution functions, one could introduce instead more realistic Maxwellians at a fixed temperature. For such a new model, an interesting connection between H-theorem and Clausius inequality can be foreseen [12].

**ACKNOWLEDGMENTS**

Work performed in the frame of the activities sponsored by MURST, and by the University of Parma, through the National Research Project "Mathematical Problems of Kinetic Theories". Helpful discussions with G. Spiga are gratefully acknowledged.

**REFERENCES**