The equilibrium and oscillations of dust grains in a flowing plasma

Sergey V. Vladimirov and Neil F. Cramer

Department of Theoretical Physics
School of Physics, The University of Sydney, NSW 2006, Australia

Abstract. The equilibrium, stability, and trapping of dust particles associated with vertical motions in the plasma sheath and pre-sheath are calculated, taking into account the dependence of the variables (such as dust charge, ion flow velocity, etc.) on the local position in the sheath/pre-sheath region of a collisional plasma with an ionization source. It is demonstrated that an increase of the rate of ionization considerably influences the equilibrium positions of dust grains, shifting them towards the electrode as well as increasing the maximum possible (equilibrium) levitation grain size.

INTRODUCTION

The structures formed by charged dust grains in a low-temperature weakly ionized plasma have attracted considerable recent interest, associated primarily with the first experiments on “dust-plasma crystals” [1] as well as later with other self-organized formations such as dust clouds, “drops”, “voids”, etc. [2–7]. Under typical laboratory conditions, dust particles are negatively charged and usually levitate in the sheath or pre-sheath region under the balance of gravitational, electrostatic (due to the sheath electric field) and plasma (such as the ion drag) forces. The ion flow, in addition to a direct (dragging) influence which can be one of the major forces supporting the formation of dust voids [4,5], is also responsible for the generation of associated collective plasma processes which can strongly affect the vertical arrangement of the grains, not only in the case of supersonic flows when a wake field is generated [8–11], but also in the case of subsonic velocities of plasma ions [12,13]. On the other hand, molecular dynamic simulations [14] have clearly demonstrated a sequence of phase transitions associated with vertical arrangements of dust grains when the strength of the confining (in the vertical dimension) parabolic potential is changed.

The first step in any research on properties of dust in a plasma is the adequate description of the surrounding plasma. In general, there are two situations of interest: (i) the dust particles do not affect significantly the properties of the plasma they are embedded in (this usually corresponds to low number densities of the dust component, i.e., to a lower number of dust particles), and (ii) the dust component is relatively dense, thus changing significantly the field and density distributions of the surrounding plasma. In this paper, we consider the case of essentially isolated dust grains (the intergrain distance exceeds the plasma Debye length), with a low total number of dust particles. Thus the first step in our study is the modelling of those plasma regions where we expect the dust particles to be trapped, the sheath and pre-sheath regions of the discharge.

Due to relatively high neutral gas pressures (often more than 50 mTorr for typical dust-plasma experiments), the laboratory plasma is strongly influenced by the effects of ion-neutral collisions. Thus the simplest mathematical approach relevant for collisionless plasmas [15] is not fully appropriate in this case. On the other hand, the correct description of collisional effects involves the speed of the ion flow and therefore naturally depends on the properties of the region (sheath or pre-sheath) we are interested in. While in the sheath region, where the speed of the ion flow is expected to exceed the ion sound velocity, a simple approximation [16] describing ion-neutral collisions can be used, in the total pre-sheath/sheath region more sophisticated approaches are necessary [17,18]. In this paper, we use an advanced model of momentum transfer between the ion and neutral species, which describes ion-neutral collisions on the basis of kinetic theory, without semi-empirical approximations [19].
Vertical oscillations of the dust can lead to the disruption of the equilibrium position of the grains. Note that most of the previous analytical models considering vertical lattice vibrations [20,21], as well as numerical models studying phase transitions [14] in the dust-plasma system, dealt with dust grains of a constant charge. In this paper, we study the whole range of possible velocities of the ion flow, treating the sheath problem self-consistently and investigating possible dust trapping as well as the disruption of the equilibrium which may occur at various positions corresponding to not only supersonic, but also subsonic ion flow velocities at the position of the dust grain in a collisional plasma with an ionization source.

THE PLASMA MODEL

The plasma consists of electrons and singly charged ions, with a uniform background neutral gas. The dust grains are assumed to have no effect on the sheath and presheath structure. We consider the one-dimensional configuration: the electrode, which is supposed to be at the constant potential -7 V, is located at the origin of the reference frame; the end of the simulation volume is at \( z_0 \) outside the sheath. All variables of interest are functions of the distance \( z \), namely, the sheath potential \( \varphi(z) \) and the electric field \( \mathbf{E}(z) = -d\varphi(z)/dz \), the speed \( v_i(z) = n_i(z) \) and density \( n_i(z) \) of the ion flow, and the electron density \( n_e(z) \) which is supposed to be Boltzmann distributed,

\[
n_e(z) = n_0 \exp\left(\frac{e\varphi(z)}{T_e}\right),
\]

where \( e \) is the electron charge, \( n_0 \) is the electron (as well as ion) number density in the plasma bulk \( (z \geq z_0) \), and \( T_e \) is the electron temperature in energy units (such that the Boltzmann constant is unity). Note that for simplicity we assume that the electron temperature is constant in the whole region of interest.

The sheath potential is determined by Poisson’s equation which, using (1), we write as

\[
\frac{d^2\varphi(z)}{dz^2} = 4\pi e n_0 \left[ \exp\left(\frac{e\varphi(z)}{T_e}\right) - \frac{n_i(z)}{n_0} \right].
\]

In this model, we neglect the total charge contributed by the dust grains (i.e., we assume the dust number density to be small).

The ion dynamics is governed by the continuity and momentum equations. The continuity equation for the ions takes into account plasma production; the main mechanism of ionization is assumed to be electron impact ionization, so that the continuity equation is

\[
\frac{d}{dz} [n_i(z)v_i(z)] = \nu_{\text{ion}} n_e(z).
\]

Here, \( \nu_{\text{ion}} \) is the plasma ionization frequency, which is proportional to the neutral gas density and varies exponentially with the inverse of \( T_e \), as well as depending on the atomic parameters of the neutral gas [23]; for argon gas it has the form

\[
\nu_{\text{ion}} = 5 \times 10^{-8} n_n \exp(-15.8/T_e),
\]

where \( \nu_{\text{ion}} \) is independent of \( z \) since we assume \( T_e \) and the neutral gas density to be uniform; \( n_n \) is measured in cm\(^{-3}\) and \( T_e \) in eV. Below, we assume that the neutral background density is kept constant, so that the variation in the ionization frequency is connected with the change in the electron temperature (see Table 1).

The momentum equation for the plasma ions is written as

\[
m_i v_i(z) \frac{dv_i(z)}{dz} = -e \frac{d\varphi(z)}{dz} - F_{\text{coll}}(z),
\]

where \( F_{\text{coll}}(z) \) is the momentum transfer rate between ions and neutrals, and the main mechanism for the ion-neutral collisions is considered to be charge exchange. Calculations on the basis of plasma kinetic theory, which allow for ion speeds comparable to the ion thermal speed, give the following expression for \( F_{\text{coll}} \) [19]:

\[
F_{\text{coll}} = \nu_E f(\delta)v_i.
\]
**Definition**

Electron temperature

Electron bulk density

Neutral density

Neutral temperature

Ion temperature

Ion mass (argon)

Electron Debye length

Ion Debye length

Ion plasma frequency

Ionization frequency

Collision frequency

**Notation**

\( T_e \)

\( n_0 \)

\( n_n \)

\( T_n \)

\( T_i \)

\( m_i \)

\( \lambda_{De} \)

\( \lambda_{Di} \)

\( \omega_{pi} \)

\( \nu_{ion} \)

\( \nu_{ei} \)

\( \nu_{e} \)

Value

1.6; 2.0; 2.9 eV

\( 10^9 \) cm\(^{-3} \)

\( 3 \times 10^{15} \) cm\(^{-3} \)

0.025

0.025

40 \times 1836 \times m_e \)

297.3; 332.5; 400.3 pm

37.17 pm

6.58 \times 10^6 \) s\(^{-1} \)

0.107 \times \omega_{pi} \)

**TABLE 1.** The main plasma parameters in the numerical computation.

<table>
<thead>
<tr>
<th>Ionization frequency</th>
<th>End of simulation</th>
<th>Electric potential</th>
<th>Electric field</th>
<th>Ion flow velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu_{ion}/\omega_{pi} )</td>
<td>( z = z_0 ) (in ( \lambda_{D_i} ))</td>
<td>( e \varphi/T_i ) at ( z = z_0 )</td>
<td>( -d_z(e \varphi/T_i) ) at ( z = z_0 )</td>
<td>( v_i/v_{Ti} ) at ( z = z_0 )</td>
</tr>
<tr>
<td>0.001</td>
<td>813.03</td>
<td>-0.0001</td>
<td>-0.0001</td>
<td>0.25</td>
</tr>
<tr>
<td>0.01</td>
<td>294.56</td>
<td>-0.0001</td>
<td>-0.0001</td>
<td>0.25</td>
</tr>
<tr>
<td>0.1</td>
<td>113.32</td>
<td>-0.0001</td>
<td>-0.0001</td>
<td>0.25</td>
</tr>
</tbody>
</table>

**TABLE 2.** The boundary conditions for the numerical computation.

Here \( \delta = |v_i|/\sqrt{2v_{Ti}(1 + T_n/T_i)^{1/2}} \), \( \nu_E \) is the average charge exchange collision frequency

\[
\nu_E = \frac{8}{3} \sqrt{\frac{2}{\pi}} q_E n_n v_{Ti} \left( 1 + \frac{T_n}{T_i} \right)^{1/2},
\]

and \( q_E \) is the characteristic charge exchange momentum transfer cross section which is practically constant and equals approximately \( 3 \times 10^{-15} \) cm\(^2\) [24] over the range of ion energies from 0.1 to 2 eV, which is of the most interest for us here. Furthermore, \( n_n \) in (7) is the density of the neutral gas (argon), \( v_{Ti} = (T_i/m_i)^{1/2} \) is the ion thermal velocity, and \( T_{i(n)} \) is the ion (neutral) temperature. In our consideration, we assume \( n_n, T_i, \) and \( T_n \) to be uniform, i.e., independent of the position \( z \) with respect to the electrode. Finally, the function \( f(\delta) \) in (6) is given by, [19],

\[
f(\delta) = \frac{3}{8 \delta^3} \left[ \left( \frac{\delta^3}{2} \right)^{-\delta^2} + \frac{\sqrt{\pi}}{\delta} \left( \frac{\delta^3}{4} \right) \text{Erf}(\delta) \right].
\]

For small \( \delta \) (low ion speeds), \( F_{coll} \) is proportional to the ion speed, while for large \( \delta \) (high ion speed), the approximation made in Ref. [16], \( F_{coll} \) is proportional to the square of the ion speed. The latter case applies in the sheath region in the calculations reported here, but not necessarily in the plasma bulk region.

Assuming the electrode has a potential of -7 V, typical of dust plasma experiments, Eqs. (2), (3), and (5) are numerically integrated to give the dependence of the potential, and thence of the sheath electric field, on \( z \). Table 1 gives the values of the plasma parameters chosen in our numerical simulations (as well as some characteristic values which are calculated on their basis, i.e., the electron/ion Debye lengths \( \lambda_{De(i)} = (T_{e(i)}/4\pi n_0 e^2)^{1/2} \) and the ion plasma frequency \( \omega_{pi} = (4\pi n_0 e^2/m_i)^{1/2} \) in the plasma bulk). The boundary conditions are chosen as those given in Table 2; note that we also have \( n_e(z_0) = n_i(z_0) = n_0 \).

**TABLE 3.** The characteristic numbers for Fig.1.
FIGURE 1. The dependence of the plasma potential $\varphi(z)$ (in V) and the electric field $E(z)$ (in V/cm), on the distance $h = z/\lambda_{Di}$ from the electrode. Figures (a1)-(a2) correspond to $\nu_{ion}/\omega_{pi} = 0.1$, figures (b1)-(b2) correspond to $\nu_{ion}/\omega_{pi} = 0.01$, and figures (c1)-(c2) correspond to $\nu_{ion}/\omega_{pi} = 0.001$. The plasma parameters are presented in Table 1. The characteristic numbers relevant to these figures are given in Table 3.

The results for the electrostatic potential and for the electric field are given in Fig. 1, where we present three sets of figures corresponding to different ratios of the ionization frequency to the ion plasma frequency $\nu_{ion}/\omega_{pi}$, viz. 0.1, 0.01, and 0.001, respectively. The first case corresponds to highest $T_e$ and input power. The other characteristic numbers, relevant for Fig. 2, are given in Table 3.

The dependences of the velocity of the ion flow $v_i(z)$ and the ion flux $n_i(z)v_i(z)$ on the distance from the electrode, found from Eq. (5), for the potential and field distributions of Fig. 1, are presented in Fig. 2. The corresponding numbers are summarized in Table 4. We see that for the region where the ion velocity Mach number (relative to the ion sound velocity $v_s$) exceeds unity (this corresponds, for the chosen plasma parameters, to $v_i > v_s = 10v_T$), the ion flux is almost constant. Moreover, comparing with Fig. 1, we see that this region corresponds to an almost linear electric field dependence on the distance from the electrode, resembling the collisionless sheath model (which we discussed in the previous paper [22]) where the region of almost linear dependence of the electric field is, however, limited to somewhat higher velocities of the flow.

FIGURE 2. The dependence of the velocity $v_i(z)$ (in units of $v_T$) of the ion flow and the ion flux $n_i(z)v_i(z)$ (in units of $n_0v_T$) on the distance $h = z/\lambda_{Di}$ from the electrode. Figures (a1)-(a2) correspond to $\nu_{ion}/\omega_{pi} = 0.1$, figures (b1)-(b2) correspond to $\nu_{ion}/\omega_{pi} = 0.01$, and figures (c1)-(c2) correspond to $\nu_{ion}/\omega_{pi} = 0.001$. The characteristic numbers relevant to these figures are given in Table 4.
THE CHARGE AND EQUILIBRIUM OF A DUST GRAIN

The charge \( Q \) of the dust particles (which is dependent on the plasma parameters, in particular, on the local electric sheath potential and the velocity of the ion flow) is found from the condition of zero total plasma current onto the grain surface:

\[
I(Q) = I_e(Q) + I_i(Q) = 0.
\]  

Note that since we are interested in collective processes on the time scale of the characteristic frequencies (of order a few times \( 10 \, \text{s}^{-1} \)), which are much lower than the charging frequency [25] (which can be of order \( 10^5 \, \text{s}^{-1} \)), we assume that (re)charging of dust grains is practically instantaneous, and we therefore neglect the charging dynamics. We note, however, that the latter can be important when considering possible instabilities of the grain levitation associated with the charging dynamics [26].

The electron and ion currents onto the dust grain are defined by

\[
I_\alpha(Q) = \sum_\alpha \int e_\alpha f_\alpha(\nu, Q) \nu d\nu.
\]

Here, the subscript \( \alpha = e, i \) stands for electrons or ions, \( e_\alpha \) and \( f_\alpha \) are the charge and distribution function of the plasma particles, with \( e_\alpha = -e_i \equiv -e, \nu \equiv |\nu| \) is the absolute value of the particle speed \( \nu \), and \( \sigma_\alpha \) is the charging cross-section which in the orbit-limited-motion (OLM) approximation is given by [27]

\[
\sigma_\alpha = \pi a^2 \left( 1 - \frac{2e_\alpha Q}{am_\alpha v^2} \right) \quad \text{if} \quad \frac{2e_\alpha Q}{am_\alpha v^2} < 1,
\]

\[
\sigma_\alpha = 0 \quad \text{if} \quad \frac{2e_\alpha Q}{am_\alpha v^2} \geq 1,
\]

where \( a \) is the radius of the dust particle and \( m_\alpha \) is the electron or ion mass. The last inequality in (11) gives a restriction on the electron charging velocities for negatively charged dust particles or for the plasma (positive) ions for positively charged dust grains.

Since the electrons are assumed to be Boltzmann distributed (Eq. (1)), the electron current (for a negative charge on the dust grain) is given by

\[
I_e(Q) = -\sqrt{8\pi} e a^2 n_0 \sqrt{\frac{T_e}{m_e}} \exp \left[ \frac{eQ(z)}{aT_e} + \frac{e\varphi(z)}{T_e} \right],
\]

where \( \varphi(z) \) is the external plasma potential at the position of the dust grain. Note that we neglect possible changes of the electron temperature in the plasma sheath.

In contrast to the electron distribution, we consider shifted Maxwellian ions with the distribution function \( f_i \propto \exp[-(v_i - \nu_i(z))^2/v_{Ti}^2] \). The inter-grain distance is assumed to be not less than the (ion) Debye length, so that the ion trajectory is affected by only a single grain. The ion current onto the dust grain in this case can be approximated by

\[
I_i = \pi a^2 e n_i(z) \bar{\nu}_i(z) \left[ 1 - \frac{2eQ(z)}{am_\alpha \bar{v}_i^2(z)} \right],
\]

where \( \bar{\nu}_i(z) = \sqrt{\nu_i^2(z) + 8v_{Ti}^2/\pi} \) [28]. Thus the charge of a (negatively charged) dust particle is determined by Eq. (9), i.e. by the equation

\[
I(Q) = I_e(Q) + I_i(Q) = 0.
\]
where \( v_e^2 = T_e/m_e \) is the squared electron thermal speed. We note that from (14) the charge can become zero for a strong enough sheath potential, such that the ion current dominates. This means that the dust particle cannot levitate and must fall onto the electrode (see the discussion in Ref. [22]). Numerical solutions of Eq. (14) for the charge of a dust particle, as a function of the particle position \( z \), are presented in Fig. 3, and the corresponding parameter values are summarized in Table 5. Here the profiles of \( \phi(z) \), \( v_i(z) \) and \( n_i(z) \) found in Section II have been used. It is apparent from Fig. 3 that the higher is the input power (i.e. the higher

<table>
<thead>
<tr>
<th>Ionization frequency</th>
<th>Charge at the electrode ( q_0 ) ( \times 10^3 )e</th>
<th>Maximum possible charge ( q_{\text{max}} ) ( \times 10^3 )e</th>
<th>Position of ( q_{\text{max}} ) ( z_{\text{qm}} ) (in ( \lambda_{\text{Di}} ))</th>
<th>Charge at the position ( v_i = v_s ) ( q_{\text{vs}} ) ( \times 10^3 )e</th>
<th>Charge at the position ( v_i = v_{i\text{f}} ) ( q_{\text{f}} ) ( \times 10^3 )e</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu_{\text{ion}}/\omega_{\text{pi}} )</td>
<td>( q_0 )</td>
<td>( q_{\text{max}} )</td>
<td>( z_{\text{qm}} )</td>
<td>( q_{\text{vs}} )</td>
<td>( q_{\text{f}} )</td>
</tr>
<tr>
<td>0.1</td>
<td>(-)</td>
<td>27.19</td>
<td>56.77</td>
<td>24.84</td>
<td>20.11</td>
</tr>
<tr>
<td>0.01</td>
<td>1.64</td>
<td>18.96</td>
<td>67.77</td>
<td>16.46</td>
<td>14.72</td>
</tr>
<tr>
<td>0.001</td>
<td>3.46</td>
<td>14.91</td>
<td>83.60</td>
<td>11.31</td>
<td>12.14</td>
</tr>
</tbody>
</table>

TABLE 5. The characteristic numbers for the dust charge calculation (Fig.3).

For a particle levitating in the sheath field, the force acting on the grain includes the sheath electrostatic force, the ion drag force, and gravity;

\[
F(z) = Q(z)E(z) - F_{\text{idr}}(z) - mg,
\]

where the ion drag force \( F_{\text{idr}}(z) = F_{\text{idr}}^{\text{col}}(z) + F_{\text{idr}}^{\text{orb}}(z) \) includes two components [28,29]: the collection force \( F_{\text{idr}}^{\text{col}}(z) \) and the orbit force \( F_{\text{idr}}^{\text{orb}}(z) \). The collection force is associated with the dust charging process, and in the OLM approximation can be written as

\[
F_{\text{idr}}^{\text{col}}(z) = \frac{4\pi e^2}{\lambda_{\text{D}}} \left[ 1 - \frac{2eQ(z)}{am_i v_i^2(z)} \right].
\]

The orbit force, which corresponds to the momentum transfer during the Coulomb collision, is given by

\[
F_{\text{idr}}^{\text{orb}}(z) = 4\pi e^2 Q^2(z) \frac{n_i(z)v_i(z)}{m_i v_i^3(z)} \Lambda,
\]

where \( \Lambda \approx \ln(\lambda_D/a) \) is the Coulomb logarithm, and \( \lambda_D \) is the plasma Debye length.
Note that the force (15) includes the \( z \)-dependence of the grain charge \( Q \), since we assume instantaneous transfer of charge onto and off the dust grain at any grain position in the sheath, such that Eq. (14) is always satisfied. The balance of forces in the vertical direction is

\[
Q(z)E(z) = m_d g + F_{i,dr}(z).
\] (18)

Solution of this equation together with the charging equation (14) gives the dependence of the charge of the grain, levitating in the sheath electric field, as a function of its size. For the levitating dust particle, there is a one-to-one correspondence of its size to its equilibrium position of levitation in the sheath, as shown in Fig. 4.

Here, we have also plotted the lines corresponding to various sizes of dust grains: as an example, \( a = 4 \mu m \), as well as the maximum possible sizes, and the sizes corresponding to a grain levitating at the position where the Mach number of the ion flow is unity (i.e., at \( v_i = v_s \)). Note that there are no equilibrium solutions for \( a > a_{\text{max}} \), the latter being a function of the ionization rate (see Table 6). The absence of an equilibrium means that the particles with such sizes will fall down onto the electrode.

**DISCUSSION AND CONCLUSION**

It is instructive to find the total potential energy, relative to the electrode position, of a single dust particle of given size at the position \( z \) in the sheath electric field:

\[
U_{\text{tot}}(z) = -\int_0^z dz' [Q(z')E(z') - F_{i,dr}(z') - m_d g].
\] (19)

Note that the total energy in this case contains not only the electrostatic energy \( Q(z)\varphi(z) \), but also terms associated with \( dQ/d\varphi \) which represent, because of the openness of the system, the work of external forces which change the dust charge. The dependence of the total potential energy on the distance from the electrode is shown in Fig. 5.

We see that the potential has a maximum and a minimum, corresponding to the two equilibrium positions found in section III. The minimum (the stable equilibrium) disappears if \( a > a_{\text{max}} \) (the curve 1) in Fig. 6. Other effects which have been neglected here, such as an electron temperature increasing towards the electrode, may serve to increase the negative charge on a grain, and so preserve an equilibrium. The critical (maximum possible for levitation) radius appears also in Fig. 5; for decreasing ionization rate, \( a_{\text{max}} \) also decreases (see
FIGURE 5. The total interaction energy $U_{tot}$ as a function of the distance $h = z/\lambda_D$ from the electrode for the different sizes of a dust particle and the different ionization rates: (a) $\nu_{ion}/\omega_p = 0.1$, (b) $\nu_{ion}/\omega_p = 0.01$, and (c) $\nu_{ion}/\omega_p = 0.001$. The curves correspond to: 1) $a = a_{max} + 1\mu m$; 2) $a = a_{max}$; 3) $a = a_{eq}$ and 4) $a = 4\mu m$. See also Fig. 5 and Table 6.

Table 6). For $a > a_{max}$, the minimum of the potential energy curve disappears thus indicating that there is no equilibrium position for such grain sizes.

Thus for a collisional plasma with an ionization source, for a grain size $a$ less than the critical radius $a_{max}$, there is a stable equilibrium position close to (or in) the presheath; for sufficiently high input powers (within a certain range of grain sizes, see Fig. 4), there can also be an unstable equilibrium position deeper inside the sheath. For $a$ greater than the critical radius $a_{max}$, there is no equilibrium position. Note that possible vertical oscillations about the stable equilibrium may develop high amplitudes, thus leading to a fall of the oscillating grain onto the electrode when the potential barrier (see Fig. 5) is overcome.

REFERENCES