Abstract. While rarefied gas dynamics has traditionally assumed a dilute gas, whose densities are so low that only binary collisions and single-body gas surface interactions occur, expressions for many-body collision rates and for many-body gas surface interaction (GSI) rates seem to suggest that at lower heights the dilute gas is not valid. In particular, in the pure rarefied regime, two-body GSIs and some three-body interactions occur whereas, in the transition regime into continuum flow, four body collisions and four-body GSIs occur. In this paper I formulate the problem of many-body energy exchange as a complement to the problem of many body scattering which I formulated in Part I. I showed there that a many body collision can be formulated in terms of the motion of a number of reduced particles which scatter in space as well as exchange energy with one another and with the centre of mass. In this paper I concentrate on the energy exchange and present a statistical theory based on reciprocity or detailed balance. I derive statistical transformation models for three and four body encounters. The same techniques can be applied to any number of interacting bodies – the final transformation model simply becomes more complex. The determination of the post-collision energies of the reduced particles makes it possible to calculate their post-collision speeds. This is needed, in combination with the scattering results, to calculate the laboratory velocities of the interacting particles. The approach produces results for use in the Direct Simulation Monte Carlo (DSMC) method, which is the standard method of computing rarefied gas phenomena.

I. INTRODUCTION

Rarefied gas dynamics has traditionally been founded on the dilute gas assumption, which presupposes that the densities are so low that only binary collisions and single-body gas surface interactions occur. However, expressions for many-body collision rates and for many-body gas surface interaction (GSI) rates seem to suggest that at lesser heights the dilute gas assumption is not valid. In fact, it appears that many body interactions become significant at lower levels (Figure 1, Ref. 1).

In Part I (Ref. 2) I looked at how to calculate the scattering component of many body collisions. I showed that many body collisions consist of the motion of a number of reduced particles each of which scatters in three dimensional space as well as interchange energy with the other reduced particles and with the centre of mass of the system. Part II completes the analysis by investigating the energy exchange among the reduced particles and the centre of mass. I propose a statistical theory based on reciprocity or detailed balance. The approach derives statistical transformations that relate the energy states at entry and exit from the collision. By calculating the post-collision energies we can find the post-collision speeds of the reduced particles, hence the laboratory co-ordinates of the particles. The analysis generates models for use in Direct Simulation Monte Carlo computations.
II. STATISTICAL MODELLING OF THE ENERGY EXCHANGE

I formulate the reciprocity equation in detail for three and four interacting bodies and then I generalise the equation to any number of interacting bodies. However, the transformation model for each number of interacting bodies must be derived individually for each case using the simple techniques illustrated for three and four body collisions. It is not possible to write down a general expression for the \(N\)-body transformation model.

Adopt the following notation for the state of the system:

\[
\mathbf{C} = (c_0, c_1, c_2)
\]

where \(c\) denotes velocity

Let the non-normalised three-particle distribution be:

\[
F_n^{(3)}(C, r)
\]

where \(r\) is the position co-ordinate

The change from initial to final states will be formulated in terms of a transition probability or interaction kernel \(R\) per unit time such that the probability of a forward encounter in the time interval \(t\) to \(t + dt\) is:

\[
R(C|C^*)dC'dt
\]

and the probability of the inverse encounter is:

\[
R(C^*|C')dC'dt
\]

Consider the statistics in an elementary volume \(dT = dx \, dy \, dz\) of physical space. Formulating the problem within this control volume makes the formulation compatible with formulations of the Boltzmann equation. If the number of molecules in this volume is \(N_v\), then the number of points in the three-particle phase space is:

\[
N_3 = \frac{1}{6} N_v (N_v - 1)(N_v - 2)
\]

This number has been calculated by taking triplets of molecules, whether or not they result in an encounter. The number of such triplets that find themselves in an elementary extension \(dC\) is:

\[
F_n^{(3)}(C)dCdr
\]

The number of these points that make a forward encounter in the time interval \(t + dt\) is:

\[
F_n^{(3)}(C')dC'dr' \times R(C'|C^*)dC'dt
\]

and the number of points that make the inverse encounter in the same time interval is:

\[
F_n^{(3)}(C^*)dC''dr'' \times R(C''|C')dC'dt
\]

At equilibrium, the reciprocity principle requires that the rates of forward and inverse encounters equal. This means equating (2) to (4). Denoting the equilibrium distribution by subscript \(e\) and introducing the velocity distribution function, \(f\), we get:

\[
f_e^{(3)}(C')R(C'|C^*)dC'dC'' = f_e^{(3)}(C^*)R(C''|C')dC'dC''
\]

Invoke Liouville’s theorem in the form of the conservation of extension in phase, i.e.

\[
dC' = dC''
\]
In the centre of mass frame we write:

\[ dC = dc_0 dc_1 dc_2 = dG G_1 G_2 dG_1 G_2 = G^2 G_1^2 G_2^2 dG dG_1 dG_2 d\Omega_1 d\Omega_2 \]

where \( d\Omega = \sin \theta d\theta d\phi \) is the elementary solid angle.

Now introduce the differential cross section \( s \) per unit solid angle for scattering into the solid angle \( d\Omega \) centred on the deflected relative velocity. We can express the transition probability in terms of this cross section as follows:

\[
\begin{align*}
R(C,C')d\mathcal{C} &= R(G',g'_1,g'_2|G,g_1,g_2) dG' G'_1 G'_2 dG G_1 G_2 = g'_1 g'_2 s' R_8(G',g'_1,g'_2|G,g_1,g_2) d\Omega'_1 d\Omega'_2 \\
R(C',C)d\mathcal{C}' &= R(G',g'_1,g'_2|G,g_1,g_2) dG' G'_1 G'_2 dG G_1 G_2 = g_1 g_2 s R_8(G',g'_1,g'_2|G,g_1,g_2) d\Omega_1 d\Omega_2 \\
\end{align*}
\]

In the first equation the RHS gives the probability of reduced particles with pre-collision relative speeds \( g_1, g_2 \) having their directions deflected into the elementary solid angles \( dW_{1f} \) and \( dW_{2f} \) centred on the post-collision relative speeds \( g_1f \) and \( g_2f \). In the second equation the RHS gives the probability of reduced particles with pre-collision relative speeds \( g_1f \) and \( g_2f \) having their directions deflected into the elementary solid angles \( dW_{1e} \) and \( dW_{2e} \) centred on the post-collision relative speeds \( g_1e \) and \( g_2e \). One describes the forward process and the other describes the inverse process. The relative speed transition probability represents the interaction among the reduced particles and the centre of mass.

If we substitute equation (9) into the reciprocity equation (7) we find that the product \( s_1 dW_{1f} s_2 dW_{2f} \) appears on the LHS whereas the product \( s_1 dW_{1e} s_2 dW_{2e} \) appears on the RHS. Inasmuch as the scattering process is correlated to the translational energies, we seek a relationship between these scattering products in terms of the translational energies. For this we consult Vincenti and Kruger (Ref. 3) who show that for inverse power molecules, with potential:

\[ V(r) = ar^{-h} \]

where \( a \) is a constant and \( h \) is the index of repulsion

the product:

\[ g^{4/h} s d\Omega \]

depends only on the dimensionless impact parameter \( \nu \), where:

\[ \nu = b \left( \frac{mg^2}{2ah} \right)^{1/h} \]

and \( b \) is the impact parameter. For the forward and inverse processes we would expect that the dimensionless impact parameters are the same, i.e. \( \nu = \nu f \). It follows that:

\[
\begin{align*}
 g^{4/h} s d\Omega' = g^{4/h} s' d\Omega' \quad \text{or} \quad \frac{s}{s'} d\Omega' = \left( \frac{g'}{g} \right)^{2/h} = \left( \frac{\nu'}{\nu} \right)^{2/h} \\
\end{align*}
\]

The use of this expression in the reciprocity equation implies that we are assuming the scattering to be correlated to the translational energies according to an inverse power law. This also means that the scattering dynamics is
correlated to the energy exchange through an inverse power law. Such correlation refines our energy exchange
models because it implies that the energy exchange is not totally uncoupled from the scattering dynamics.

The translational energy in the centre of mass system is:

\[ e_t = \frac{1}{2} m^3 g^2 \Rightarrow de_t = m^3 g^2 dg = K e_t^{1/2} de_t, \quad e_G = \frac{1}{2} MG^2 \Rightarrow de_G = MGdG = B e_G^{1/2} de_G \]

where \( m^3 \) is the three-body reduced mass, and \( K \) and \( B \) are constants.

The equilibrium distribution is:

\[ f_e^{(3)}(C) = A \exp \left( -\frac{\Theta_G + e_I + e_{I2}}{kT} \right) \]

Note that in writing the equilibrium distribution in this form we have assumed the many-body kinetic energy
expression derived in Ref. 2.

Now substitute all these intermediate results into the reciprocity equation of (7) and focus attention on the energy
exchange components. Do not forget to include the extra \( g \) term from equation (9) \(^1\)

Writing:

\[ R_e \left( G', g', g_1, g_2 | G, g, g_1, g_2 \right) = R_e \left( e_I', e_{I1}', e_{I2}' | e_I, e_{I1}, e_{I2} \right) \quad \text{and} \quad e_I = e_I' + e_{I1} + e_{I2} \]

and extracting the energy exchange part from the resulting reciprocity equation we get:

\[ e_I' g_{e1}^{-1} e_{I1}' g_{e1}^{-1} e_{I2}' g_{e1}^{-1} \exp \left( -\frac{e_I' + e_{I1}' + e_{I2}'}{kT} \right) \quad \text{and} \quad e_I = e_I' + e_{I1} + e_{I2} \]

where \( g_I = \frac{3}{2} \), \( g_0 = 2 - \frac{2}{h} \)

This is the final form of the energy reciprocity equation that we are after. What remains is to construct
appropriate forms for the transition probability or interaction kernel. In so doing we must recognise the constraints
imposed by the DSMC method, which relies on statistical sampling to compute the behaviour of individual
molecules. Thus, we must symmetrise this reciprocity equation so that the left and right hand sides take identical
forms. We do so by transforming our view from interaction kernels to correlation densities with associated
statistical transformations. The correlation densities correlate the pre-collision and post-collision states using the
statistical transformations, which become the required models of the interaction. This view is particularly suited to
DSMC thinking. The mathematical backing, for carrying out this analysis in terms of correlation densities and
statistical transformations, derives from distribution calculus – the theory of continuous probability distributions.
Three important theorems from the calculus are particularly pertinent. They are the transformation, contraction and
extension theorems. We have already exploited some of their tenets in the foregoing derivations.

### III. SYMMETRISING THE RECIPROCITY EQUATION

To symmetrise the reciprocity equation, we introduce a set of correlation variates \( s \) with correlation densities \( G(s) \),
expressible in the form:

\[ G(s) = \prod G_i(s_i) \quad \text{and satisfying the symmetry constraint } G' = G'' = G \]
The correlation variates are used to relate the pre- and post-collision states through a set of transformation equations. To give a closed system of transformation equations we introduce as many equations for the correlation variates as are needed. Thus, the transformation equations convert an extended pre-collision state to an extended post-collision state. This extended representation derives from including the correlation variates as part of the state representation. Remember that the correlation variates are dummy variables whose transformation equations are ultimately discarded, when we use the model in numerical simulations or DSMC computations.

In terms of this extended representation the collision is epitomised by the mapping:

\[(S', s') \to (S'', s'') \quad \text{or} \quad (S'', s'') = \Psi(S', s')\]

where \(S\) denotes the energy state of the system. To preserve the symmetry property of the extended reciprocity equation the transformation is defined such that it has the same form as its inverse, i.e.

\[\Psi = \Psi^{-1} \quad \text{or} \quad (S', s') = \Psi(S'', s'')\]

This property is important in developing transformations that satisfy the reciprocity equation, i.e. that transform the left hand side to the right hand side and vice versa.

In general, we wish to relate the pre- and post-collision distributions in the contracted space in the manner:

\[g'(S') \to g''(S')\]

where \(S\) represents the energy state of the system. Instead we extend the state to \(S, s\), by introducing the dummy variates \(s\), and relate the extended distributions in the manner:

\[g'(S', s') \to g''(S'', s'') \quad \text{or} \quad g''(S'', s'') = J(\Psi^{-1})g'(S', s')\]

where \(J(\Psi^{-1})\) is the Jacobian of \(\Psi^{-1}\), and \(\Psi\) is the overall transformation constituting the collision model. In this expression, we have:

\[S' = S'(S'', s''), \quad s' = s'(S', s'')\]

These relationships define the inverse overall transformation. In practice, to use (13), we write:

\[g'(S', s') = g'(S')G(s'), \quad g''(S', s') = \int \int \int \int s''g''(S'', s'')ds'' = \int \int \int s''J(\Psi^{-1})g'(S', s')ds'\]

where \(\vec{a}\) is the model parameter vector and \(G(s)\) are the correlation densities.

Note that equation (14) is the average of the post-collision extended distribution taken over the dummy correlation variates. This averaging process introduces the functional dependence of the post-collision distribution on the model parameters.

In a complex non-equilibrium gas, the pre-collision distribution will be unknown. Even if it were, performing the necessary integrations in (14) would be highly complex. However, in a DSMC computation we usually know the state of the molecules before encounter. This can be supplied to the transformation model for use in computing the post-collision state. This is all that we want.

Now introduce the dummy variates and their densities into the contracted reciprocity equation by writing:

\[R_e(\vec{e}_{G,1}, \vec{e}_{G,2}|\vec{e}_{G,1}^e, \vec{e}_{G,2}^e)de_1^e de_2^e ds^e = G(s')ds', \quad R_e(\vec{e}_{G,1}, \vec{e}_{G,2}|\vec{e}_{G,1}^e, \vec{e}_{G,2}^e)de_1^e de_2^e ds^e = G(s')ds'\]

The extended reciprocity equation reads:

\[e_1^e g_{a_1}^{-1} e_2^e g_{a_2}^{-1} \exp\left(-\frac{\vec{e}_G^2 + \vec{e}_{a_1}^2 + \vec{e}_{a_2}^2}{kT}\right)G(s')ds' de_1^e de_2^e = \int R_e(\vec{e}_{G,1}, \vec{e}_{G,2}|\vec{e}_{G,1}^e, \vec{e}_{G,2}^e)de_1^e de_2^e\]

The extended reciprocity equation reads:

\[\frac{\vec{e}_G^2 + \vec{e}_{a_1}^2 + \vec{e}_{a_2}^2}{kT}\]

\[= e_1^e g_{a_1}^{-1} e_2^e g_{a_2}^{-1} \exp\left(-\frac{\vec{e}_G^2 + \vec{e}_{a_1}^2 + \vec{e}_{a_2}^2}{kT}\right)G(s')ds' de_1^e de_2^e\]

\[\text{(15)}\]
For each energy mode write:

\[ x = \frac{\theta}{kT} \]

and introduce the Gamma distribution for \( x \) with parameter \( m \):

\[ \text{Ga}(x|m) = \frac{1}{\Gamma(m)} x^{m-1} e^{-x}, \quad (m > 0, \quad 0 < x < \infty) \]

where \( \Gamma(m) \) is the gamma function.

The extended reciprocity equation becomes:

\[ \text{Ga}(x'|g_0) \text{Ga}(x'_1|g_1) \text{Ga}(x'_2|g_2) G(s') dx'_1 dx'_2 = \text{Ga}(x'_0|g_0) \text{Ga}(x'_1|g_1) \text{Ga}(x'_2|g_2) G(s) dx'_0 dx'_1 dx'_2 \]  \hspace{1cm} (16)

This reciprocity equation looks simple and symmetrical, and is highly tractable using distribution calculus – through exploitation of the properties of Beta and Gamma distributions. The equation can be readily generalised to any number of interacting bodies. For \( N \) interacting bodies the reciprocity equation is:

\[ \text{Ga}(x'_0|g_0) dx'_0 \prod_{i=1}^{N-1} \text{Ga}(x'_i|g_i) dx'_i \ G(s') ds' = \text{Ga}(x'_0|g_0) dx'_0 \prod_{i=1}^{N-1} \text{Ga}(x'_i|g_i) dx'_i \ G(s) ds'' \]  \hspace{1cm} (17)

The problem of solving our many body collision system has now degenerated into one of formulating the correlation densities \( G \). We do so by formulating a set of transformation equations, which constitute our model, to convert the pre-collision state to the post-collision state and vice versa. The overall transformation model is formulated by judiciously putting together a number of elementary transformations derived from distribution calculus. As we construct the overall transformation the correlation densities are generated automatically as part of the mathematics. Both transformation model and correlation densities are symmetrical with respect to the forward and inverse processes; i.e. the same transformation equations and correlation densities perform the forward and inverse processes. This means that the forward and inverse transformations have identical forms, and that the forward and inverse correlation densities are identical.

I illustrate the construction procedure using three and four bodies, with and without an external force acting on the system. The simple procedure can be applied to any number of interacting bodies but each case must be handled separately. It is not possible to write down a general expression for the model transformation and correlation densities for an \( N \)-body system.

IV. CONSTRUCTING THE STATISTICAL MODEL: THREE BODY COLLISIONS

We consider two cases: interactions under an external force and interactions with no external force. To construct the statistical transformations we split each component energy into active and inactive components. Both the total active and total inactive energies are conserved in the interaction. Thus, we split the pre-collision energies into active and inactive components and then recombine the active components into the total active pre-collision energy. This total pre-collision energy is then redistributed into post-collision active components, which are finally recombined with their pre-collision inactive counterparts to give the final post-collision energies. The whole point is that only the active components are interchanged in the interaction, i.e. are redistributed among the reduced particles. Inactive components are not. This means that the post-collision inactive components are equal to their pre-collision inactive counterparts.

No External Force

In this case the centre of mass moves with constant velocity. Therefore we only concern ourselves with the reduced particles. The LHS of the reciprocity equation is:

\[ dF = \text{Ga}(x'_0|g_0) \text{Ga}(x'_1|g_1) \text{Ga}(x'_2|g_2) G(s') dx'_1 dx'_2 \]  \hspace{1cm} (18)
Start by choosing:

\[ G(s') = b(s'|a,g_r,(1-a_i)g_i) b(s'_i|a,g_r,(1-a_i)g_i) \]  
where \( 0 \leq a_i \leq 1 \) is a model parameter. \( (19) \)

Decompose the energies into active and inactive parts by introducing the transformations:

\[ \rho'^1 = \rho'^1_{1a} + \rho'^1_{12}, \quad s'_1 = \frac{\rho'^1_{1a}}{\rho'^1_{1a} + \rho'^1_{12}} \quad \text{with inverse} \quad \rho'^1_{1a} = s'_1 \rho'^1_{1}, \quad \rho'^1_{12} = (1-s'_1) \rho'^1_{1} \]

\[ \rho'^2 = \rho'^2_{1a} + \rho'^2_{12}, \quad s'_2 = \frac{\rho'^2_{1a}}{\rho'^2_{1a} + \rho'^2_{12}} \quad \text{with inverse} \quad \rho'^2_{1a} = s'_2 \rho'^2_{1}, \quad \rho'^2_{12} = (1-s'_2) \rho'^2_{1} \]

The LHS of the reciprocity equation becomes:

\[ dF = G(\rho'^1_{1a} | a,g_r) G(\rho'^1_{12} | (1-a_i)g_i) b(s'_i|a,g_r,(1-a_i)g_i) \]  
\[ \cdot d\rho'^1_{1a} d\rho'^1_{12} d\rho'^1_{1a} d\rho'^1_{12} \]  
\[ \cdot d\rho'^2_{1a} d\rho'^2_{12} d\rho'^2_{1a} d\rho'^2_{12} \]  
\[ \cdot dP' \quad (20) \]

Now recombine the active parts into the total active energy by introducing the transformation:

\[ x_a = \rho'^1_{1a} + \rho'^1_{2a}, \quad s_a = \frac{\rho'^1_{1a}}{\rho'^1_{1a} + \rho'^1_{2a}} \quad \text{with inverse} \quad \rho'^1_{1a} = s_a x_a, \quad \rho'^1_{2a} = (1-s_a) x_a \]

The LHS of the reciprocity equation becomes:

\[ dF = G(x_a | a,g_r) G(\rho'^1_{12} | (1-a_i)g_i) b(s'_i|a,g_r,(1-a_i)g_i) \]  
\[ \cdot d\rho'^1_{1a} d\rho'^1_{12} d\rho'^1_{1a} d\rho'^1_{12} \]  
\[ \cdot d\rho'^2_{1a} d\rho'^2_{12} d\rho'^2_{1a} d\rho'^2_{12} \]  
\[ \cdot dP' \quad (21) \]

A new correlation density has been generated. This is:

\[ G = b(s'_i|a,g_r,(1-a_i)g_i) \]

Now introduce the correlation density:

\[ G = b(s'_i|a,g_r,(1-a_i)g_i) \]

and decompose the active energy into post-collision active components by writing:

\[ x_a = \rho'^1_{1a} + \rho'^1_{2a}, \quad s_a = \frac{\rho'^1_{1a}}{\rho'^1_{1a} + \rho'^1_{2a}} \quad \text{with inverse} \quad \rho'^1_{1a} = s_a x_a, \quad \rho'^1_{2a} = (1-s_a) x_a \]

The LHS of the reciprocity equation becomes:

\[ dF = G(x_a | a,g_r) G(\rho'^1_{12} | (1-a_i)g_i) b(s'_i|a,g_r,(1-a_i)g_i) \]  
\[ \cdot d\rho'^1_{1a} d\rho'^1_{12} d\rho'^1_{1a} d\rho'^1_{12} \]  
\[ \cdot d\rho'^2_{1a} d\rho'^2_{12} d\rho'^2_{1a} d\rho'^2_{12} \]  
\[ \cdot dP' \quad (22) \]

Finally, recombine the post-collision active components to the inactive components by writing:

\[ \rho'^1_{1} = \rho'^1_{1a} + \rho'^1_{12}, \quad \rho'^1_{1a} = \frac{\rho'^1_{1a}}{\rho'^1_{1a} + \rho'^1_{12}} \quad \text{with inverse} \quad \rho'^1_{1a} = s'_1 \rho'^1_{1}, \quad \rho'^1_{12} = (1-s'_1) \rho'^1_{1} \]

\[ \rho'^2_{1} = \rho'^2_{1a} + \rho'^2_{12}, \quad \rho'^2_{1a} = \frac{\rho'^2_{1a}}{\rho'^2_{1a} + \rho'^2_{12}} \quad \text{with inverse} \quad \rho'^2_{1a} = s'_2 \rho'^2_{1}, \quad \rho'^2_{12} = (1-s'_2) \rho'^2_{1} \]

The LHS of the reciprocity equation becomes:
\[ dF = \mathbf{G}(x'_{11} | g_{1}) \mathbf{x}_{12} \mathbf{G}(x'_{12} | g_{2}) G(s') ds'dx'_{11}dx'_{12} \]

where:
\[ G(s') = b(s'_{11} | a_{1}, g_{1}, (1 - a_{1}) g_{1}) b(s'_{12} | a_{2}, g_{2}, (1 - a_{2}) g_{2}) b(s'_{12} | a_{2}, a_{2}, g_{1}) b(s'_{11} | a_{1}, a_{1}, g_{2}) \]

We see that the LHS of the reciprocity equation has been transformed into the RHS thus confirming that the sequence of elementary transformations satisfies reciprocity. Combining now the elementary transformations we get the following model of the energy exchange:

\[ x'_{11} = (1 - s'_{11}) x_{11} + s'_{11} x_{10}, \quad x'_{12} = (1 - s'_{12}) x_{12} + (1 - s'_{10}) x_{10} \]

where \( x_{10} = s'_{11} x_{11} + s'_{12} x_{12} + s'_{10} x_{10} = \text{total active energy} \).

The required correlation densities are:

\[ G(s') = b(s'_{11} | a_{1}, g_{1}, (1 - a_{1}) g_{1}) b(s'_{12} | a_{2}, g_{2}, (1 - a_{2}) g_{2}) b(s'_{12} | a_{2}, a_{2}, g_{1}) b(s'_{11} | a_{1}, a_{1}, g_{2}) \]

Note that we have ignored transformation equations for the post-collision correlation variates, inasmuch as we do not need these in numerical computations. By adding the first two of equation (24a) we can show that the overall transformation satisfies energy conservation. We can also combine the elementary transformations in reverse to get the pre-collision state expressed in terms of the post-collision state. This gives the inverse process. Checking the form of this inverse transformation we find that the required symmetry property is satisfied with respect to the forward and inverse process.

With External Force

In this case the centre of mass accelerates through the collision and must be included in the collisional exchange. The LHS of the reciprocity equation is:

\[ dF = \mathbf{G}(x'_{11} | g_{1}) \mathbf{x}_{12} \mathbf{G}(x'_{12} | g_{2}) G(s') ds'dx'_{11}dx'_{12} \]

Following the approach illustrated above we obtain the final transformation model as:

\[ x'_{11} = (1 - s'_{11}) x_{11} + s'_{11} x_{10}, \quad x'_{12} = (1 - s'_{12}) x_{12} + (1 - s'_{10}) x_{10} \]

where \( x_{10} = s'_{11} x_{11} + s'_{12} x_{12} + s'_{10} x_{10} = \text{total active energy} \).

The correlation densities are:

\[ G(s') = b(s'_{11} | a_{1}, g_{1}, (1 - a_{1}) g_{1}) b(s'_{12} | a_{2}, g_{2}, (1 - a_{2}) g_{2}) b(s'_{12} | a_{2}, a_{2}, g_{1}) b(s'_{11} | a_{1}, a_{1}, g_{2}) \]

Note that in addition to ignoring the transformation equations for the post-collision correlation variates we have also ignored the transformation equation for the post-collision centre of mass speed. This is because we have chosen to determine the motion of the centre of mass dynamically, using equation (3) of Ref. 2.

V. CONSTRUCTING THE STATISTICAL MODEL: FOUR BODY COLLISIONS

No External Force

The LHS of the reciprocity equation is:

\[ dF = \mathbf{G}(x'_{11} | g_{1}) \mathbf{x}_{12} \mathbf{G}(x'_{12} | g_{2}) G(s') ds'dx'_{11}dx'_{12} \]

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Again we follow a similar approach to the one illustrated earlier to get the transformation model as:

\[ x_1' = (1 - s_1) x'_1 + s_1' s_2' a_1, \quad x_2' = (1 - s_2') x'_2 + (1 - s_2) s_3' a_2, \quad x_3' = (1 - s_3') x'_3 + (1 - s_3) x'_d \]  

where \( x_a = s_1' x'_1 + s_2' x'_2 + s_3' x'_3 = s_1^2 x'_1 + s_2^2 x'_2 + s_3^2 x'_3 \) is the total active energy.

The correlation densities are:

\[ G(s') = b(s'_1 | a, g_1, (1 - a) g_1) b(s'_2 | a, g_2, (1 - a) g_2) b(s'_3 | a, g_3, (1 - a) g_3) b(s'_d | a, g_d, a, g_d) \]  

\[ (28a) \]

**With External Force**

The correlation densities are:

\[ G(s') = b(s'_1 | a, g_1, (1 - a) g_1) b(s'_2 | a, g_2, (1 - a) g_2) b(s'_3 | a, g_3, (1 - a) g_3) b(s'_d | a, g_d, a, g_d) \]  

\[ (28b) \]

The LHS of the reciprocity equation is:

\[ dF = \int G(s') G(x'_1 | g_c f x_f c f ) G(x'_2 | g_f c a f ) G(x'_3 | g_a f ) G(x'_d | g_f c a f ) d(s') d(x'_1) d(x'_2) d(x'_3) \]  

\[ (29) \]

Proceeding as illustrated earlier we get the overall transformation as:

\[ x_1' = (1 - s_1) x'_1 + s_1' s_2' a_1, \quad x_2' = (1 - s_2') x'_2 + (1 - s_2) s_3' a_2, \quad x_3' = (1 - s_3') x'_3 + (1 - s_3) x'_d \]  

where \( x_a = s_1' x'_1 + s_2' x'_2 + s_3' x'_3 + s_1^2 x'_1 + s_2^2 x'_2 + s_3^2 x'_3 \) is the total active energy.

The required correlation densities are:

\[ G(s') = b(s'_1 | a, g_1, (1 - a) g_1) b(s'_2 | a, g_2, (1 - a) g_2) b(s'_3 | a, g_3, (1 - a) g_3) b(s'_d | a, g_d, a, g_d) \] \times b(s'_d | a, g_d, (1 - a) g_d) b(s'_d | a, g_d, a, g_d) b(s'_d | a, g_d, a, g_d) b(s'_d | a, g_d, a, g_d) \]  

\[ (30a) \]

**VI. MODEL PARAMETER DETERMINATION AND NUMERICAL SIMULATIONS**

The single model parameter in the exchange models can be determined by matching experimental data for the temperature dependence of a chosen transport coefficient, such as the thermal conductivity of a gas. For astronomical applications the parameter must be chosen to match some experimental data in celestial mechanics.

To perform a calculation for a given collision we sample the required correlation variates from the correlation densities. We then substitute these into the model equations together with the pre-collision molecular states. Evaluating the RHS of the model equations gives the post-collision energies. These are used to determine the post-collision speeds of the reduced particles. For the centre of mass we determine the motion dynamically. To perform computations we need to sample from the Beta distribution. See Phillips and Beightler (Ref. 4) for Beta distribution sampling.

**IX. REFERENCES**

1 Agbormbai A. A., Collision Rates for Many Body Encounters, submitted to Rarefied Gas Dynamics, 22nd symp.