Relativistic Electromagnetic Solitons
Produced by Ultrastrong Laser Pulses
in Plasmas

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Abstract. Low frequency, relativistic sub-cycle localised (soliton-like) concentrations of the electromagnetic (em) energy are found in two-dimensional (2D) and in three-dimensional (3D) Particle in Cell simulations of the interaction of ultra-short, high-intensity laser pulses with homogeneous and inhomogeneous plasmas. These solitons consist of electron and ion density depressions and intense em field concentrations with a frequency definitely lower than that of the laser pulse. The downshift of the pulse frequency, due to the depletion of the pulse energy, causes a significant portion of the pulse em energy to become trapped as solitons, slowly propagating inside the plasma. In an earlier phase solitons are formed due to the trapping of the em radiation inside an electron cavity, while ions can be assumed to remain at rest. Later on, after $(m_i/m_e)^{1/2}$ times the laser period, ions start to move and the ion depletion occurs producing a slowly growing hole in the plasma density. In inhomogeneous plasmas the solitons are accelerated toward the plasma vacuum interface where they radiate away their energy in the form of bursts of low frequency em radiation. In the frame of a 1D cold hydrodynamic model for an electron-ion plasma, the existence of multipeaked em solitons has been investigated both analytically and numerically. The analytical expression for a sub-cycle relativistic soliton has been derived for circularly polarized pulses in a cold isotropic plasma, and in the presence of an externally applied magnetic field. Recently, em relativistic solitons in a hot multi-component plasma have been investigated in the frame of an hydrodynamic (adiabatic) model and of a kinetic (isothermal) model.

An overview of the most recent analytical and numerical results on the soliton dynamics is given.
INTRODUCTION

The rapid development of the laser technology occurred during the last decade [1] has made available table-top laser sources of femtosecond pulses with intensities up to $10^{21}$ W/cm$^2$, which allow one to investigate the radiation-matter interaction under extreme conditions characterised by the electron momentum largely exceeding $m_e c^2$ [2]. Moreover, the exciting perspectives of getting much higher laser intensities in matter in a relatively near future [3], have stimulated a renewed interest towards the physics of ultrarelativistically intense radiation in matter, and in particularly in plasmas. Two-dimensional (2D) and three-dimensional (3D) Particle-in-Cell (PIC) numerical simulation of the interaction of an ultrashort ultraintense laser pulse with a preformed plasma show that a large number of nonlinear processes take place, as for example, laser frequency variation, high-order harmonic generation, the appearance of coherent nonlinear structures (plasma channels, relativistic solitons, vortices), the generation of ultraintense quasi-static electric and magnetic fields, electron and ion acceleration to relativistic energies. Up-to-date reviews of such effects can be found in Refs. 4 and 5. Generally speaking, a non-negligible fraction of the laser pulse energy is involved in these processes, then it is of primary importance to investigate their nature in order to control their occurrence and possibly to take advantage from their “applications”. In this paper we shall focus our attention onto the relativistic electromagnetic solitons (RES), which play an important role in the strongly nonlinear interaction between electromagnetic (EM) radiation and plasmas, and we shall review the most recent theoretical results on this subject. Here, we shall refer to RES as to the localised distributions of the EM field, characterized by a normalized value of the transverse electric field $a = eE/m_e c \omega > 1$ (here, $e$ and $m_e$ are the electric charge and the rest mass of an electron, $c$ is the speed of light in vacuum, $E$ and $\omega$ are the transverse component of the electric field and the frequency of the EM radiation, respectively), either manifesting a finite drift velocity or being stationary; in both cases, they are long-living EM structures trapped within quasistationary electron density hollows. In a cold plasma, they are the result of the equilibrium of two forces: the ponderomotive force, originated from the nonuniform distribution of the EM energy, and the electrostatic force, associated with the charge separation. We wish to mention that EM soliton-like structures are of concern for astrophysics and cosmology, as well. Indeed, it is believed that spatial density fluctuations in the early Universe (between $10^{-2}$ and 1 sec after the Big Bang) are at the origin of the galaxies and clusters of galaxies formation [6]. In addition, spatial temperature nonuniformities of the early hot plasma would cause the observed nonuniformities in the distribution of the cosmic microwave background radiation. Therefore the interest for the physics of the RES extends well beyond the laboratory applications and deserves an accurate general analysis.

THE STATUS OF THE RESEARCH BEFORE PIC SIMULATIONS

Let us begin with a short summary of the theoretical studies on RES performed till the end of the nineties. Several analytical investigations dealing with the interaction of the EM radiation of relativistic amplitude with plasmas have been undertaken in the past.
following the pioneering paper by Akhiezer and Polovin [7]. There, the relativistic nonlinear fluid equations for cold electrons in a uniform positive background, coupled with the Maxwell equations, where used to describe transverse, longitudinal and coupled modes of arbitrary amplitude. In later times, the problem of the propagation of EM radiation of relativistic amplitude in an overdense plasma attracted a lot of attention for its potential use in dense plasma heating [8,9]. Standing wave structures of large amplitude (although, non relativistic) at the boundary with an overdense plasma, accompanied by plasma density bunching, were described by analytical [10] and numerical [11] approaches, allowing movable ions. Exact relativistically intense optical standing waves were considered for the first time in Ref. 12, in the frame of the fluid model developed in Ref. 7. In this paper there are all the ingredients to describe one-dimensional circularly-polarized RES, with fixed ions and the cold electron fluid. One-dimensional (1D) circularly-polarized solitons in an electron-ion plasma have been investigated in Ref. 13, in the quasi-neutral approximation, that is by assuming that the charge unbalance is small if compared with the density of each plasma component. The full problem of 1D circularly-polarized RES in a cold electron-ion plasma has been tackled by Kozlov, Litvak and Suvorov [14]. They have derived a new set of two nonlinear coupled equations for the scalar amplitude of the vector potential and for the electrostatic potential, which allowed to investigate non uniform spatial distributions of the fields (the equations developed in Ref. 7 refer to waves with constant amplitude). Localised solutions in the form of drifting solitons, accompanied by an electron density depression and by an ion density concentration, have been obtained by numerical integration of the two relevant second order differential equations. Multi-humped concentrations of the EM energy (either with even or odd number of nodes, \(p\)), inside electron density cavities bounded by density "walls", have been found in Refs. 14 and 15 (see also Ref. 16 in the weakly relativistic case with fixed ions). They can be interpreted as due to the trapping of light inside a self-generated electrostatic wave, which is excited at the front of the moving soliton and absorbed at its rear. The asymptotic equilibrium between the relativistic ponderomotive force and the electrostatic field induced by the charge separation leads to a concentration of the ion density in correspondence of the electron depression. It is worth mentioning that in Ref. 14, it has been claimed that i) within the cold fluid approximation, circularly-polarized one-dimensional solitons can exist only if they move at a velocity larger than a critical one, defined as \(c(m_e/m_i)^{1/2}\) (\(m_i\) is the ion rest mass); ii) the quasi-neutral approximation, which assumes a negligible charge separation to occur, applies to small amplitude drifting solitons only. The nature of localised distributions of EM energy propagating at a speed close to \(c\) has been exploited in Refs. 15 and 17, with the aim of exploiting the potentialities to accelerate charged particles to relativistic velocities. In particular, the possibility of using a so-called "triple-soliton" (a combination of two EM and one electrostatic waves in the beat-wave resonance), propagating in a plasma channel, has been investigated in Ref. 17. The main advantage of using a soliton drifting almost at light speed, instead of a laser pulse as a driver for accelerating particles, is that it does not produce any wakefield behind itself, thus overcoming the problem of the pump depletion, so severe in the classical laser-driven accelerator concepts [18].

All the above mentioned investigations dealing with fully relativistic field amplitudes have been based on cold fluid models, with mobile or fixed ions, under quasi-neutral
approximation or with the charge separation taken into account. In general it is possible
to neglect a finite plasma temperature whenever the velocity acquired by the electrons
under the action of the electric field is much larger than the electron thermal velocity.
Generally speaking, this condition is well satisfied in laboratory experiments for values
of the dimensionless parameter $a$ of the order or larger than unity. However there are a
number of reasons urging one to consider RES in warm plasmas. A first example is the
following. A cold plasma model leads to density spatial distributions which may become
negative for a particular choice of the soliton parameter values. This is due to the fact that
if the full plasma cavitation is achieved, in the vacuum region fluid equations are
meaningless and one should solve the field equations piece-wise, in the plasma region
and in the vacuum region separately, and then match properly at the boundaries (see the
recent studies on wave penetration into an overdense plasma in Refs. 19-21). This
procedure can be somewhat elaborated and produces consistent density profiles with
discontinuous derivatives and very spiky spatial distributions (in these cases their
stability can be seriously compromised). The introduction of a finite electron temperature
changes the structure of the longitudinal (that is, in the direction of wave propagation)
component of the electron motion equation, leading to a non-negative electron density
(this can be easily verified if one assumes a Boltzmann equilibrium distribution for the
electron density [22]). Moreover, the inclusion of a finite temperature means that the
soliton is now the result of the equilibrium of three forces: the ponderomotive force, the
electrostatic force, and the thermal force. A second example of the importance of
including a finite plasma temperature in the fluid theories of RES is found in cosmology.
As we have already mentioned, the formation of RES in the primordial plasma could be
an important source of large-scale density and temperature nonuniformities which are at
the origin of the galaxies and clusters of galaxies formation [6]. In particular, it is
conjectured that in the early epoch of the Universe evolution the matter was in the form
of a mixture of electrons, positrons, and photons in thermal equilibrium at a temperature
larger than $m_e c^2$ [6]. It is evident that any problem of propagation of relativistic EM
waves in such a hot environment should be formulated in the frame of a hot-plasma
model. It is worth noticing that to this aim a proper set of hot-plasma hydrodynamic
equations should be derived which treat the thermal motion and the ordered dynamics of
the fluid elements consistently from the point of view of the special relativity. Generally
speaking, the assumption of a Boltzmann distribution for the particle density in the
presence of a ponderomotive and an electrostatic potential [22] represents a model which
allows simple analytical calculations, but is devoid of any physical background unless
both nonrelativistic field amplitudes and plasma temperatures are considered.
With reference to RES in electron-positron plasmas, we wish to mention few theoretical
results. The existence of one-dimensional solitons in a cold magnetised electron-positron
plasma has been demonstrated in Ref. 23 and 24. It was argued that in a cold electron-
positron plasma no solitons could be produced since no charge separation is expected in
the case of equal inertia of the plasma constituents. RES solitons have then been
investigated in a cold isotropic electron-positron-ion plasma, where a small fraction of
heavy ions causes the appearance of an electrostatic potential distribution which is
sufficient to balance the effect of the ponderomotive potential nonuniformity [25]. The
case of a hot electron-positron-ion plasma was also studied in Ref. 26, while the hot
electron-positron plasma was considered in Refs. 27-29. The above analysis have made
use of the slowly varying envelope approximation in order to simplify the Maxwell
equations for the EM field trapped in the form of a soliton [24-29]. Interesting
discussions on the modelling of a hot electron-positron plasma in the presence of
relativistically intense EM radiation can be found in Refs. 30-32.
The above review of the published literature on RES, being by no means exhaustive,
shows the deep interest manifested by theoreticians for the subject. In the next Sections
we shall describe the most recent analytical and numerical results on RES produced in the
wakefield of an ultrastrong laser pulse in a plasma, achieved by the wide international
collaboration to which the authors belong.

LOW-FREQUENCY SUB-CYCLE RES IN NUMERICAL SIMULATIONS

Since the early investigations on the laser wakefield acceleration of electrons [33], it was
clear that during its propagation through an uniform underdense plasma, the laser pulse
undergoes dramatic changes in its characteristics, i.e. shape, frequency, and intensity, due
to the strongly nonlinear character of the interaction. In particular, it was observed that
the laser pulse frequency and amplitude both change during the interaction in such a way
as to preserve the total photon number: that is, the process is adiabatic [34]. 1D [34-36]
and 2D [37-39] PIC numerical simulations, with mobile electrons and fixed ions, agree
on the wave dynamics: during its propagation the laser pulse is strongly depleted both by
the electrostatic and magnetostatic wakefield excitation (magnetic vortex formation was
studied in Ref. 40; for a relativistic hydrodynamic study of the wakefield excitation see
also Ref. 41) and by the stimulated (backward) Raman scattering; therefore its amplitude
decreases and, due to the invariance of the total number of photons, its frequency and its
group velocity should also decrease. A characteristic propagation length for the nonlinear
depletion can be estimated as $\ell_{\text{depl}} = c\ell_{\text{depl}} = \ell_p (\omega/\omega_p)^2$ for a narrow laser beam, and as
$\ell_{\text{depl}} = \ell_p (\omega/\omega_p)^2$ in the case of a broad laser pulse [34,42], where $\ell_p$ is the pulse length.
Therefore, portions of the initially propagating radiation ($\omega > \omega_p$) slow down and
become trapped ($\omega < \omega_p$) inside localised density depressions, drifting with a group
velocity which is appreciably smaller than the laser pulse speed. Notice that 1D
simulations show a finite drift velocity of the soliton (which may be even negative) [35],
while 2D simulations predict an almost zero soliton speed in a uniform background
plasma [37-39]. The fundamental role played by the lowering of the laser frequency in
the soliton formation is supported by the strong spectral broadening [34] and frequency
down-shifting (up to a factor 0.25 of the laser frequency) [39] characterising the EM
radiation in the regions of the solitons.

PIC simulations have shown that RES are produced in a broad range of physical
conditions of the laser-plasma interaction: for linearly and circularly polarized laser
radiation, for both s- and p-polarized pulses, for unperturbed plasma densities varying in
the range from 0.04 to $2n_c$ (where the critical density is defined as $n_c = m_e\omega^2/(4\pi e^2)$), for
initial normalized pulse amplitudes $\alpha$ in the range from 1 to 10. More than 20 % of the
laser pulse EM energy is trapped in the form of solitons and does not contribute to the
wakefield acceleration process. RES observed in the simulations have no high frequency
component, i.e. they are sub-cycle solitons, containing half a wavelength of the EM field. As a consequence, they cannot be described in terms of an envelope but the full field structure should be retained in any theoretical approach. Once they have been formed, RES are long-living EM/plasma structures. In particular, numerical simulations of a plasma with mobile electrons and fixed ions show that a single soliton achieves an almost stationary state within a hundred laser radiation cycles. However, in particular conditions solitons are able to interact with the background plasma and between each other in a quite peculiar way. 1D simulations show that, for a proper choice of the laser-plasma interaction conditions, two solitons can be generated which move along trajectories intersecting each other: the two solitons undergo a “collision”. After this event, the two solitons emerge presenting the same characteristics they had before their approaching. 2D simulations show that RES are generated with almost zero group velocity in a uniform plasma. If two solitons of opposite phases are produced close to each other, in the course of time their fields tends to oscillate in phase, after that the two solitons merge into a single soliton which preserve the total EM energy [38,43]. A 2D RES can be made to move if it is produced in a nonuniform plasma [39]. In a weakly nonuniform dispersive plasma the soliton moves following the equations of geometric optics: 

\[
\frac{dr}{dt} = \frac{\partial K}{\partial k}, \\
\frac{dk}{dt} = - \frac{\partial K}{\partial r},
\]

where the Hamiltonian is the wave frequency \( K(r,k) = \left( k^2 c^2 + \omega_{pe}^2 \right)^{1/2} \). In particular its drift occurs along the direction opposite to the electron density gradient. Then, if a RES is generated in a finite plasma, with its density decreasing towards its boundaries, the soliton is bound to escape from the plasma region and, once at the edge, it leaves the plasma emitting its EM energy in the form of a burst of radiation. The spectrum of the radiation manifests a wide peak well below the laser frequency [39]. The experimental observation of the frequency spectra of the radiation emitted from a plasma used as a target for ultrastrong laser pulses could reveal the onset of physical processes causing the laser frequency down-shift [44]: it could be an indirect proof of the soliton formation.

For dimensionless laser pulse amplitudes \( a \gg 1 \) or for long simulations times (longer than \( (m_i/m_e)^{1/2} \) the radiation period) the ion dynamics begins to play an important role in the strong laser-plasma interaction and specifically in the dynamics of RES. This is due to the fact that both the electrostatic field associated with the charge separation where the soliton is localised, and the ponderomotive force act in order to redistribute the ion density. Recent 2D [45-47] and 3D [48-50] PIC simulations with mobile electrons and ions show that the asymptotic soliton dynamics is rich of new physics. The soliton formation, that is the laser frequency down-shift followed by the trapping of part of the EM energy inside an electron cavity, while the ion density remains unperturbed, is similar to what has been observed previously. Indeed, it takes place in a time much shorter than the ion response time ( \( = 2\pi (m_i/m_e)^{1/2} \omega_{pe}^{-1} = 2\pi \omega_{pe}^{-1} \)). Later on, the Coulomb repulsion in the electrically non neutral ion core pushes the ions away, in a process which is analogous to the so-called “Coulomb explosion” [51-54], resulting in an ion acceleration to typical energies of the order of \( m_e c^2 a_{max} \) [46,47]. The plasma cavity goes on expanding slowly in the radial direction. In the 2D case, the expansion can be modelled within the “snow plow” approximation [55], which for \( t \to \infty \) gives \( R \propto t^{1/3} \). A characteristic time scale of the cavity expansion is \( \tau = (6\pi R_0^2 n_0 m_e / (k^2 c^2))^{1/2} \) where \( R_0 = c/\omega_{pe} \) is the soliton radius [45]. The total EM energy trapped in the cavity is constant and then the field amplitude and the frequency decrease as \( E \propto t^{-4/5} \) and \( \omega \propto t^{-2/5} \), respectively.
This entity which behaves as a soliton in the early part of its evolution, has been named “post-soliton” in the ion-dominated phase, which is no longer a stationary state.

The soliton formation and their subsequent evolution into a post-soliton have been observed in 3D PIC simulations, as well [48-50]. The EM structure of the 3D soliton is such that the electric field is poloidal and the magnetic field is toroidal. Therefore it has been named TM-soliton. The soliton core manifests an average positive charge, resulting in its Coulomb explosion and in the ion acceleration. Asymptotically, the quasi-neutral plasma cavity is subject to a slow continuous radial expansion, while the soliton amplitude decreases and the ion temperature increases.

**ANALYTICAL STUDIES OF LOW-FREQUENCY SUB-CYCLE RES**

Several analytical investigations of the RES have been undertaken after it has become clear from multi-dimensional numerical simulations that such coherent EM structures not only play an important role in the energetic balance of the laser-plasma interaction, but in principle can be controlled and used for charged particle acceleration. The exact 1D analytical solution of the cold fluid equations for the electrons with fixed ions, for arbitrary intense circularly polarized EM radiation is given in Ref. 56. It corresponds to a single-hump, non drifting RES in an overdense plasma. The relationship between the peak vector potential amplitude \( a_0 \) and the radiation frequency \( \omega \) of the soliton is \( a_0=\frac{2e_0\omega}{\omega_c^2-\omega^2}\omega^{-2} \). Its characteristic width is \( \approx \sqrt{c/\left((\omega_c^2-\omega^2)/2\right)} \) and the minimum electron density \( n_0 \) at the centre of the soliton is related to \( \omega \) by the equation \( n_0=n_0\left[1-4(1-\omega^2)/(\omega^2)^2\right] \), where \( n_0 \) is the unperturbed electron density. The maximum field amplitude \( a_m=3^{1/2} \) corresponds to the complete expulsion of the electrons, that is, \( \omega=\sqrt{2/3} \), and \( n_0=0 \). This class of solutions of the cold fluid equations undergoes wavebreaking for amplitudes larger than \( a_m \). Indeed, 1D PIC simulations demonstrate a long-term robustness of the analytical solution for \( a_0=1.5 < a_m \). On the contrary, a soliton-like structure with an initial amplitude \( a_0=3 \) breaks and its energy is converted into the kinetic energy of fast electrons up to several \( m_ec^2 \). The analysis of Ref. 56 has been extended to the case of a soliton in a uniform magnetised plasma [57]. If the radiation is circularly polarized, due to the inverse Faraday effect [58-60] a magnetic field normal to the polarization plane is produced and it may affect the soliton properties. The 1D cold fluid equations for the magnetised plasma electrons, coupled with the Maxwell equations, have been reduced to a single nonlinear second order ordinary differential equation for the normalized vector potential amplitude \( a \). We wish to notice that i) the generalised momentum conservation writes \( (1-\Omega/\omega)\gamma p=\gamma a \), where \( \Omega=eB/m_ec \) is the electron cyclotron frequency, and \( p=p_e/m_ec \), is the normalised electron momentum; ii) the relativistic factor \( \gamma=(1-v^2/c^2)^{1/2} \), once written in terms of the vector potential, satisfies the algebraic equation \( (\gamma^2-1)(\gamma-\Omega/\omega)^2-\gamma a^2=0 \); for \( B=\Omega=0 \) we recover the familiar \( \gamma=(1+a^2)^{1/2} \). Again there is a maximum allowed amplitude corresponding to the zero electron density. However, in the magnetised case this value can be controlled by the magnetic field intensity and by its sign (through the ratio \( \Omega/\omega \)). For negative values of \( \Omega/\omega \), corresponding to radiation frequencies lower than those allowed for \( B=0 \), solitons with amplitudes appreciably larger than in the unmagnetised case exist, if \( |\Omega/\omega| \sim O(1) \).
The previous investigations [14,15] on multi-humped, moving one-dimensional solitons in a plasma with movable electrons and ions have been reconsidered and new results on the consequences of the ion motion on the soliton structure have been found [61,62]. By numerically integrating the system of two second-order ordinary differential equations for the vector potential amplitude and the electrostatic potential, an extensive study of the RES and of the corresponding electron and ion density spatial distributions has been performed. The study of the spectra of the multi-humped solitons \((p \geq 1)\), that is their existence condition in the frequency-velocity \(\omega (1-V^2/c^2)/\omega_{pe, V/c}^2\) plane, shows that i) for velocities smaller than a critical value \(V_{bf}^c\) (where a bifurcation occurs), no solution can be found; an important consequence is that the results found in Ref. 56 for non drifting \((V=0)\) solitons are not continuously connected with those of the case \(V \neq 0\). ii) For \(V_{hif} < V < V_{br}\) two solitary waves are found: the upper solution is that of the fixed-ion case, slightly modified by the ion dynamics, while the lower branch appears only if the ion dynamics is taken into account; iii) for \(V > V_{br}\), high frequency solution is found only. Here \(V_{br}\) is the velocity of soliton breaking: for values of \(V \rightarrow V_{br}^+\) on the low frequency branch, field and density spatial distributions peak and a singularity appears at \(V_{br}\) (for \(p=1\), \(V_{br}=0.32c\), for \(p=2\), \(V_{br}=0.46\)). The ions tend to pile up at the centre of the soliton, while the electrons accumulate at its edges (the electron “walls” of Refs.14,15). A potential hump, moving at \(V=V_{br}\) is thus built up whose maximum peak value is \(e\phi_{bf}=(1-(1-V_{br}^2/c^2))^{1/2}m_{e}c^2\). When the soliton breaks, the associated electromagnetic energy is expected to be released in the form of accelerated ions whose energy can be estimated as \(E_{ion}=e\phi_{bf}(1-V_{br}^2/c^2)\), corresponding to several tens of MeV.

The models retaining the effects of a finite plasma temperature are likely to smooth out or change several critical features of the cold fluid descriptions, as for example negative density values or the onset of wavebreaking. Moreover, if we interpret the moving sub-cycle soliton as an EM wave-packet, whose front excites a relativistic plasma wave, which is subsequently absorbed by its rear, the transition from solitons travelling with a group velocity of the order of the speed of light, towards slowly moving EM structures should necessarily deal with the problem of wave excitation in a finite temperature plasma. Finally, since the temporal evolution of post-solitons results in the particle acceleration and the plasma heating [45], it is worth developing hot plasma models. This task has been pursued in two recent papers devoted to the search for RES in a multi-component plasma with arbitrary temperatures [63,64]. In the first paper [63] the set of relativistic hydrodynamic equations for a hot plasma coupled with the Maxwell’s equations have been derived from first principles, starting from the conservation laws of the particle number and of the energy-momentum tensor (the approach of Ref. 65 has been extended to a conducting fluid). The adiabatic closure of the fluid equations leads to the entropy conservation for each species. Similar equations have been derived in Ref. 66, by calculating the equations for the evolution of the moments of the relativistic Vlasov equations. There, the closure was based on the use of the relativistic Maxwellian for the calculation of the average particle energy. Subsequently, the model developed in Ref. 66 was used to investigate RES in hot plasmas [26-29-67]. It is natural to define an effective particle mass for each component \(s, m_{eff}=m_s\gamma_s R_s\), where \(\gamma_s=(1+p_s^2+\alpha_s^2/R_s^2)^{1/2}\) and \(\gamma_s=(1+p_s^2+\rho_s^2Z_s^2a_s^2/R_s^2)^{1/2}\) are the relativistic factors for electrons and ions, respectively, \(p_s=p_{\|}/m_s c^2\) is the normalised parallel particle momentum, \(\rho=m_e/m_s\) and the function \(R_s=1+\alpha_s T_e/m_s c^2\) has been introduced, where \(\alpha_s=T_s/(T_e-1)\), \(T_s\) is the adiabatic
index, and $T_e$ is the temperature. Notice that the temperature is a dependent variable of the problem, then it is not possible to get $\gamma$ in explicit form as functions of the electrostatic and the vector potentials, only. The conservation law for the transverse component of the generalised momentum for each species writes $R_p + q_A = \text{const.}
abla$.

The system has been reduced for studying 1D, circularly-polarized, non-drifting RES in an overdense electron-positron $(e,p)$ plasma of arbitrary temperature. Since the two species have equal masses and absolute values of the charge, for $T_e = T_p$, the plasma does not develop any charge separation. The existence conditions for RES have been studied for different radiation frequencies and unperturbed values of the temperatures, $T_{co}$, $s$. In a strongly overdense plasma ($\omega^2 < \omega_{pe}^2$), the ultrarelativistic kinetic pressure (the scaling $T_{co} < \omega^{-2}$ holds) and the ponderomotive pressure balance each other, giving rise to extremely large amplitude solitons. In the limit of vanishingly small temperatures ($T_{co} < m_e c^2$), even a modest trapped field amplitude ($\alpha < 1$) is able to dig a deep plasma hole up to the full plasma expulsion at the centre of the soliton. A similar analysis performed in the slowly varying approximation, in underdense electron-positron plasmas has been developed in Ref. 67.

An alternative model (isothermal) where the plasma temperature is spatially uniform has been developed by giving the particle distribution function $f_s(p_s, P_s) = \left[ N_s e^{K_s p_s^2} / \sqrt{2\pi\sigma_s^2} \right] \exp(-P_s^2 / \sigma_s^2)$, which is an exact solution of the relativistic Vlasov equation, for circularly-polarized EM radiation in a 1D plasma [64]. Here, $W_s(x,t) = m_e c^2 \gamma + q_s \phi(x,t)$ is the total particle energy, $P_s(x,t) = p_s + q_A(x,t)/c$ is the particle generalised momentum, and $\gamma = (1 + p_s^2 / m_e c^2)^{1/2}$ is the relativistic factor. $K_s$ is the modified Bessel function of first order and $\beta_s = T_e / m_e c^2$. It represents a highly anisotropic particle distribution function with a finite parallel temperature, $T_s$, and a transverse beam-like distribution in the momentum space, with a zero perpendicular energy spread. The above distribution function is used to calculate the sources of the fields in the Maxwell equations, that is, the charge and the current densities, and to obtain the closed set of two ordinary differential equations for the electrostatic potential and for the scalar amplitude of the vector potential. The case of RES in an isothermal electron-positron plasma has been investigated. Equilibria with extremely high field intensities in strongly overdense plasmas have been obtained. In contrast to the adiabatic case [63], no lower limit occurs on the temperature in order to have solitons. Moreover, the isothermal model [64] predicts the possibility of the full plasma cavitation in an extended spatial region of width $-1/\omega$, at sufficiently small plasma temperature. This model is presently being worked out to investigate RES in quasineutral electron-ion plasmas [68].

CONCLUSIVE REMARKS

In the present review we have described the main lines of the recent theoretical, analytical and numerical, researches on the relativistic electromagnetic solitons, which have had new impulse with the fast development of ultra-high intensity laser pulses and with the consequent possibility of investigating exotic regimes of the strong radiation-matter interaction, previously outside the experimental capabilities of terrestrial laboratories. Very recently [69] the first experimental evidences of the formation of long-living macroscopic bubble-like structures have become possible by means of a newly
developed diagnostic, the proton imaging [70]. It is based on the deflection of a proton beam due to the quasistatic electric field produced by the charge separation associated with the localised EM wave-packet. The dynamics of such structures is in good agreement with the multi-dimensional PIC numerical simulations of clusters of RES. These multi-dimensional kinetic codes have revealed themselves excellent tools to investigate such peculiar regimes of laser-plasma interaction.

From the point of view of the analytical studies, it would be worth to develop multi-dimensional descriptions of RES for arbitrary amplitude. Indeed, it is a formidable task, due to the strongly nonlinear characters of the relevant equations. However, the clear evidences of the quasi-neutral character of RES over long times, can be a physically well-grounded simplification of the problem. An other aspect of the analytical approaches which needs theoretical support is the study of the stability of the exact solutions. Since there are no general stability criteria for the complicated systems of equations of Refs. 14, 57, 61, 63, 64, probably a good way to test the stability of 1D solitons is to use PIC codes with the analytical solutions as initial conditions, as it was done in Ref. 56.

Finally, once the occurrence of RES will have been definitely established, the next step will be to learn how to control such extremely energetic objects and how to use them for applicative purposes.

REFERENCES

Relativistic Electromagnetic Soliton in a Collisionless Plasma”, *this Conference.*


