A COMPARISON OF ULTRASONIC WAVE REFLECTION/TRANSMISSION MODELS FROM ISOTROPIC MULTI-LAYERED STRUCTURES BY TRANSFER-MATRIX AND STIFFNESS-MATRIX RECURSIVE ALGORITHMS

Krishnan Balasubramaniam, Vikram Mukundan, M Vikram Reddy
Center for NDE, IIT Madras, MEMH/MDS 301, Chennai, 600 036, India

ABSTRACT. Two well-known algorithms for isotropic multilayer structure problem transfer matrix method and stiffness matrix method are compared in terms of numerical stability. A modification has been applied to the transfer matrix approach to enhance its stability, especially at higher frequency and thickness values. The modification proves to be stable, and the results are compared with the unconditionally stable stiffness matrix algorithm. A procedure to include the interface stiffnesses in the stiffness matrix algorithm has been discussed and verified. The stability of the algorithms for large number of layers are also presented with the other results and verifications.

INTRODUCTION

Modeling of acoustic wave propagation in layered elastic media is important because of the extensive range of problems, which can be addressed. A popular and widely reported method for modeling wave propagation in layered structures is the Transfer Matrix Method, which has been credited to Thomson [1] and has been well documented [2-5].

There are several publications, which have addressed the stability issue and suggested improvements to reduce the instability. For instance, Dunkin [5] had developed a Delta operator technique, which has since been improved by Kundu and Mal [7], Levesque and Piche [8] and Castaings and Hosten [9]. Speed of computation is even more critical during the use of inverse techniques for determining the material properties, where the procedure has been frequently repeated during a typical inverse search process [13]. Furthermore, as reported by Castaings and Hosten [9], closed form analytical expressions for the local transfer matrix elements and sub-determinants need to be found using mathematical software that performs symbolic operations. This leads to the development of considerably new formulation of the technique, which can sometimes result in very complex coding for numerical analysis for both the forward and the inverse solutions. For thick structures with periodic repetition of layers, Floquet wave techniques may be applicable [13]. A numerical approximation method has been shown to work well in several thick structures [11]. A numerical truncation algorithm was discussed by K Balasubramaniam [18] for structures with a high ultrasonic frequency, large number of layers or very high thickness.

Global matrix method was proposed by Knopoff [12] as an alternative approach to a computationally stable solution. In this method the equation for all the layers are accounted in a single matrix. A modified global matrix method using a layer stiffness matrix has been obtained for an isotropic medium by Kaussel and Rosset [15]. The method also reduces the solution to a global banded linear system of equations. Another recent approach to modeling
for ultrasonic testing of multilayered structures is to involve the stiffness matrix by Rokhlin and Wang [20].

Here we discuss comparisons of the numerical stabilities of these two approaches, transfer matrix [20] and stiffness matrix [21] along with the modified transfer matrix approach. We also incorporate the interface stiffness between the layers, widely used to simulate the adhesion between the layers, into the stiffness matrix approach. NDE methods using ultrasonic waves to evaluate the transmission and reflection phenomena at the interfaces is discussed by Segal and Rose (1980). Many authors have addressed the dependence of reflectivity on adhesive bond quality using imperfect boundary conditions in one dimensional case for normal incidence- Baik and Thompson(1984), Pilarski(1983), Tattersall(1973). This was extended to a case where the incidence takes place from a solid semi-space onto a solid layer embedded between two solids and bonded in an imperfect manner by A. Pilarski, J. L. Rose and K Balasubramaniam [19].

BACKGROUND

In the study of interaction of acoustic wave with solid multilayered structures separating two semi-infinite media. We have considered the materials to be homogeneous and isotropic. The solid planes are perpendicular to the z-axis and we consider only plane acoustic waves in the x-y plane to be incident. The stiffness constants, KN normal to the layer and KT tangential to the layer are associated with the interfaces between the solid layers.

The acoustic waves in any layer due to oblique acoustic waves can be approached either by the classical methods proposed by Thomson and Haskell or by multi-reflection model. We follow the former model. According to this model any layer has all the four types of acoustic waves viz incident transverse, incident longitudinal, reflected transverse and reflected longitudinal. The angles of propagation of all these waves are defined by Snell’s Law. For this we need to know the incident angle, transverse velocity, and longitudinal velocity of the solid layers.

The displacement at any point within the layer can be determined by the superposition of the displacements due each of the four modes described above. The displacements u, v and w in x, y, z directions respectively can be expressed in as follows, where the superscript denotes the displacement due to the upward or downward traveling wave and the subscript denotes the mode - longitudinal or transverse.

\[
\sum u = u_L^+ + u_T^+ + u_L^- + u_T^-
\]
\[
\sum v = 0
\]
\[
\sum w = w_L^+ + w_T^+ + w_L^- + w_T^-
\]

MODEL DESCRIPTION

Transfer Matrix Approach

From the earlier relations, the displacements and stress at the (j-1)th interface with the wave amplitudes at the corresponding nth layer is represented in matrix notation. Similarly for the jth interface and the same nth layer can be written. The transfer matrix between the displacement and stress values across each layer is determined by eliminating the acoustic wave amplitudes, which remains same within a layer. Thus we get the following
Where \([C]_n\) is the transfer matrix defined by
\[
[C]_n = [D]_j[D]_{j-1}^{-1}
\]  

By using the continuity of stress and appropriate boundary condition for the displacement (in terms of stress and interface stiffness), we relate the stress and displacement of any two layers as
\[
\begin{align*}
\sigma_{x_j}^+ &= C_{11} u_j^- + C_{12} w_j^- + C_{13} \tau_{x0}^- + C_{14} \sigma_{x0}^- \\
\tau_{x_j}^+ &= C_{21} u_j^- + C_{22} w_j^- + C_{23} \tau_{x0}^- + C_{24} \sigma_{x0}^- \\
\sigma_{x_j}^- &= C_{31} u_j^- + C_{32} w_j^- + C_{33} \tau_{x0}^- + C_{34} \sigma_{x0}^- \\
\tau_{x_j}^- &= C_{41} u_j^- + C_{42} w_j^- + C_{43} \tau_{x0}^- + C_{44} \sigma_{x0}^- 
\end{align*}
\]  

Thus we are able to propagate through the layers by the transfer matrix and across the interfaces by the boundary conditions. So we can relate the stress and displacement of any layer at any interface with any other.

Finally by relating the stress, displacements at the first and the last interfaces which are already known from the amplitudes, incident wave, reflected longitudinal, reflected transverse, directed longitudinal and directed transverse, the four unknowns namely RLL, RTT, DLL and DTT are determined.

**Modified Transfer Matrix Approach**

In studying the computational procedures of the transfer matrix, it was observed that one of the reasons for the computational instabilities was due to gross inconsistencies in the order of matrix values. This stems from the inherent definition of the propagator matrix in the first place. For instance in the case of the plane wave incidence, the propagator matrix is of order 4x4 with the first two rows corresponding to displacement values and the last two rows corresponding to the stress values.

Thus any matrix operation like calculation of determinant or inverse of the matrix, is hindered by the difference in order of the values, i.e. the displacement terms are negligible compared to the stress value terms. This manifests in the form of numerical instabilities at higher frequencies. A very simple modification was found to yield better stability. This was achieved by dividing the last two rows in the propagator matrix by a suitable value such as \(10^{12}\). Thus the values within the same matrix are brought in the same order. This modification was tested and proved to yield better stability to the transfer matrix approach.

The first two rows of the transfer matrix relate the displacements of one layer with that of the next layer. The last two rows relate the stresses at the layer to that of the next. It was observed that the stress values were greater than the first two rows by an order of about \(10^{12}\). Thus while undergoing matrix operations the displacement values become negligible. By scaling the last two rows we bring the entire matrix to the same order. The modification thus imparted stabilizes the matrix operations as in determinants and inverse calculations.

**Stiffness Matrix Approach**

In this model the stress, displacement relation is expressed as stiffness between the stress and displacements of any two interfaces. The following relation is expressed in terms of the displacements and stresses. We can find a relation between the displacements and stresses
across the interfaces as

\[
\begin{bmatrix}
\tau_{j+1} \\
\sigma_{j+1} \\
\tau_j \\
\sigma_j
\end{bmatrix} =
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} \\
K_{21} & K_{22} & K_{23} & K_{24} \\
K_{31} & K_{32} & K_{33} & K_{34} \\
K_{41} & K_{42} & K_{43} & K_{44}
\end{bmatrix}
\begin{bmatrix}
u_{j+1} \\
w_{j+1} \\
u_j \\
w_j
\end{bmatrix}
\]

(5)

The above theory has been discussed in detail by S. I. Rokhlin and L. Wang [20]. At this point we include the modifications to the stiffness matrix approach to incorporate the interface stiffnesses along the normal and tangential directions to the surface. These are achieved by bringing in the stiffnesses while accounting for the boundary conditions at each of the interfaces. In ultrasonic testing of layered structures one of the critical defects is the interfacial weakness. An ideal connection between two solids can be described as infinitely rigid welded boundary condition, i.e. both the tangential stiffness (KT) and the normal stiffness (KN) tend to infinity. This assumes continuity of both displacements and stresses in both the tangential and normal directions. Thus we are able to associate the reflection and transmission factors to the interfacial weaknesses through stiffness variations. Weaknesses in the interfaces can be modeled with finite values for KT and KN. The sensitivity is restricted only to an interval of KT and KN and this depends upon the layer properties.

The boundary conditions with interface stiffness can be applied as,

\[
\begin{bmatrix}
\tau_j \\
\sigma_j \\
\tau_{j+1} \\
\sigma_{j+1}
\end{bmatrix} =
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} \\
K_{21} & K_{22} & K_{23} & K_{24} \\
K_{31} & K_{32} & K_{33} & K_{34} \\
K_{41} & K_{42} & K_{43} & K_{44}
\end{bmatrix}
\begin{bmatrix}
u_j - \tau_j / KT_j \\
w_j - \sigma_j / KN_j \\
u_{j+1} \\
w_{j+1}
\end{bmatrix}
\]

(6)

Now we can eliminate the jth layer by eliminating the common displacements u and w from the above relations. Similar to the first model we can propagate from first interface to the last interface, to get the global matrix in a recursive loop to determine the unknowns. This model inherently avoids the exponential overflows by using the stiffness matrix in place of the transfer matrix.

**Model for Attenuation**

Attenuation has been incorporated in the following manner. The attenuation is represented by amplitude that decreases exponentially with distance. This results from the \( k_x \) term having a complex component. The complex component in wave velocity in the material is modeled by adding an imaginary part to the material elastic constants. This imaginary component is dependent on frequency. Thus we give the Lame’s constants as

\[
\mu = \mu + j \omega \eta_1 \\
\lambda = \lambda + j \omega \eta_2
\]

where \( \eta_1 \) and \( \eta_2 \) act as the attenuation coefficients of the material and \( \omega \) is the frequency. This results in complex velocities for longitudinal and transverse modes, which in turn produces complex wave number in the exponents. Results of this model are shown in Fig. 3(d).

**RESULTS AND OBSERVATIONS**

The numerical instability in the transfer matrix algorithm can be attributed to two reasons viz, (1) failure of the matrix operations due to the discrepancies in the order of the
values in the propagator matrix. (2) Numerical overflow due to very high exponential powers caused by high \( fd \) values, where \( f \) is the frequency and \( d \) the thickness of the layer. In the modified transfer matrix approach the first kind of instability is avoided by reducing the discrepancies in the order of matrix values encountered. In the stiffness matrix approach the second kind of instability, due to the high exponential powers, is also overcome. The stability of the stiffness matrix approach results from rearrangement, which causes the stiffness matrix itself. This rearrangement inherently avoids discrepancies in the order of values within the same matrix.

The Figs.-1(a) and (b) shows the variation of RLL with changing incident angle, by the transfer matrix method and stiffness matrix method. The instabilities in the transfer matrix method are predominantly of the first kind mentioned above. The modified transfer matrix also works well and comparable with the stiffness matrix results as shown in Fig 1(c). The Figs. 1(d) and (e) show the variation of RLL with the increasing frequency for the transfer matrix approach and stiffness matrix approach respectively. We see that while the transfer matrix approach induces numerical instabilities beyond certain frequencies, the stability of the stiffness matrix approach was verified much beyond (1000 MHz) the transfer matrix. Moreover the transfer matrix approach shows minor instabilities even for small frequency range. Further the modified transfer matrix approach yields stability comparable with the stiffness matrix approach as seen in Fig 1(f).

Figure 2(a) shows the RTT values for varying stiffness at the first interface, by the
transfer matrix approach. The modification to the Stiffness matrix approach to include stiffnesses between the interfaces was verified. Fig 2(b) shows the same result as obtained from the modified form of the Stiffness matrix approach. In these it can be seen that the sensitivity to changing stiffness values occurs around \(10^{12}-10^{16}\). In these verifications the value of KT was taken that as twice the KN value.

The stability of the algorithms was also tested for higher number of layers. The reflection factors were computed for a 10 layer structure with 8 layers (d=0.1mm) between aluminium semi-spaces. Fig 3(a) shows the instability occurring in the transfer matrix approach even at low fd values (10-11 MHz-mm). Fig 3(b) and (c) shows the stabilities of the stiffness matrix and the modified transfer matrix approaches respectively, even at very high fd values.

The instability of the modified transfer matrix approach creeps in at very large fd values. This is found to occur after a specific value of fd which implies the failure due exponential overflows. The stiffness matrix approach is however categorically stable even at very high fd values. However in practical applications in NDE we do not anticipate such high fd values(1e9 MHz-mm) except in special cases as in geotechnical applications. Thus, while the stiffness matrix approach and the modified transfer matrix are clearly much more stable than the transfer matrix approach, the stiffness matrix approach is superior among the three. But for practical applications the transfer matrix approach with the modification serves well.

The two models were also tested for computational performance in terms of time taken. The models were run to generate the RLL values for incidence angle varying from 0 to 180.

**FIGURE 2.** Computed reflection factors for an aluminium-epoxy semi-infinite spaces, for an incident angle of 30, and frequency = 50 MHz for varying normal stiffness values of the first interface, using the (a) Modified Transfer Matrix approach; (b) Stiffness Matrix Approach.

**FIGURE 3.** Computed reflection factors for a 8 layers of thickness 0.1 mm between aluminium semi-infinite media for normal incidence and varying frequencies by (a) Transfer Matrix approach; (b) Modified Transfer Matrix approach; (c) Stiffness Matrix approach (d) for a single layer steel layer of thickness 2 mm between epoxy semi-infinite media for normal incidence with material attenuation using Modified Transfer Matrix approach.
90 for a fixed frequency of 2 MHz. This was performed on a Pentium III-866 processor. The models were run for an epoxy-aluminium structure with three, ten and fifteen layers. The results are displayed in Table 1. It can be seen that the Modified Transfer Matrix method takes less time in each of the cases compared to the Stiffness Matrix approach. Moreover as the number of layers is increased, the gain in time saved is more for the Modified Transfer Matrix method. During computationally intensive calculations involving inversion of material properties the gain in time will add up to be considerable.

EXAMPLE SOLID ROCKET MOTOR BOND NDE

The application of the above theory is found in detection of adhesive disbonding in multi-layered structure. The theoretical simulations can be used to locate regions of maximum sensitivity in the frequency - angle domain for detecting these disbondings. The disbonding is

<table>
<thead>
<tr>
<th>Model</th>
<th>No. of Layers</th>
<th>Time Taken(s)</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modified Transfer matrix</td>
<td>3</td>
<td>16.01</td>
<td>4.93</td>
</tr>
<tr>
<td>Stiffness Matrix</td>
<td>3</td>
<td>16.80</td>
<td></td>
</tr>
<tr>
<td>Modified Transfer Matrix</td>
<td>10</td>
<td>33.20</td>
<td>16.39</td>
</tr>
<tr>
<td>Stiffness Matrix</td>
<td>10</td>
<td>38.64</td>
<td></td>
</tr>
<tr>
<td>Modified Transfer Matrix</td>
<td>15</td>
<td>40.92</td>
<td>31.09</td>
</tr>
<tr>
<td>Stiffness Matrix</td>
<td>15</td>
<td>53.64</td>
<td></td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th>Material</th>
<th>Density(kg/m$^3$)</th>
<th>C$_{I}$(m/s)</th>
<th>C$_{T}$(m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>7800</td>
<td>5600</td>
<td>3000</td>
</tr>
<tr>
<td>Aluminium</td>
<td>2700</td>
<td>6320</td>
<td>3080</td>
</tr>
<tr>
<td>Epoxy</td>
<td>1300</td>
<td>2800</td>
<td>1100</td>
</tr>
<tr>
<td>Polysterene</td>
<td>1050</td>
<td>2670</td>
<td>1370</td>
</tr>
<tr>
<td>Air</td>
<td>1.27</td>
<td>340</td>
<td>-</td>
</tr>
</tbody>
</table>

FIGURE 4. Intensity plots of amplitude differences at all incidence angles for frequencies 10-100 KHz (a) disbonding at 1$^{st}$ layer (b) disbonding at 2$^{nd}$ layer and for frequencies 100-1000KHz for (c) disbonding at 1$^{st}$ layer (d) disbonding at 2$^{nd}$ layer.
characterized by low values of interface stiffness across the layer. Detection is characterized
by the difference in the magnitude of the difference amplitudes between the ‘good’ and the
disbondings. This difference is plotted in the frequency – angle domain as intensity plots (Fig.
4) and the sensitive regions are located.

REFERENCES

1. Thomson W.T. et. al., @ Transmission of elastic waves through a stratified solid
2. A.H. Nayfeh.Wave Propagation in Layered Anisotropic Media with Applications to
3. B.Hosten and M.Castaings, A Transfer matrix of multilayered absorbing and anisotropic
media. Measurements and simulations of ultrasonic wave propagation through composite
4. M. Deschamps and B. Hosten, AThe effects of viscoelasticity on the reflection and
5. J. W. Dunkin, " Computation of modal solutions in layered elastic media at high
6. B. Hosten. "Bulk heterogeneous plane waves propagation through viscoelastic plates and
stratified media with large values of frequency domain." Ultrasonics Vol 29, 445-449
7. T. Kundu, and A. K. Mal, " Elastic waves in a multi-layered solid due to a dislocation
8. D. Levesque and L. Piche, AA robust transfer matrix formulation for the ultrasonic
9. M. Castaings and B. Hosten, ADelta operator technique to improve the Thomson-Haskell-
method stability for propagation in multilayered anisotropic absorbing plates, @ J. Acoust.
Soc. Am. 95, 1931(1994).
10. C. Potel, J.F. de Belleval, AA acoustic propagation in anisotropic periodic multilayered
12. L. Knopoff, A matrix method for elastic wave problems, Bull. eism. Soc. Am.54 0431-
13. H. Schmidt, G. Tango, Efficient global matrix approach to the computation of synthetic
14. H. Schmidt, F. B. Jensen, A full wave solution for propagation in stratified viscoelastic
media with application to Gaussian beam reflection at fluid/solid, J. Acoust. Soc. Am 77,
15. E. Kausal, J. Roesset, Stiffness matrices for layered soils, Bull. Seism. Soc. Am. 71 1743-
16. B. L. Kennett, Seismic Wave Propagation in Stratified Media, Cambridge University.
17. B. L. Kennet, N. J. Kerry, Seismic waves in stratified half-space, Geophys. J. R. Astron.
Soc. 57 557-583, (1979).
18. K. Balasubramaniam, On a Numerical Truncation Algorithm for Transfer Matrix Method,
Characteristics of Reflectivity From a Solid Layer Embedded Between Two Solids With
Imperfect Boundary Conditions, Journal of Acoustical Society of America, 87 (2) pp. 532-
20. L. Wang and S. I. Rokhlin, Stable reformulation of transfer matrix method for wave