PLATE-WAVE DIFFRACTION TOMOGRAPHY FOR STRUCTURAL HEALTH MONITORING

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ABSTRACT. This paper presents a tomographical reconstruction approach to quantify planar damage in plate-like structures. By adopting the Mindlin plate theory, theoretical solutions are derived for the scattering or diffraction wave fields due to a damage represented by an inhomogeneity. The relative simplicity of Mindlin's plate theory as opposed to the exact Rayleigh-Lamb solution in describing both the symmetric and anti-symmetric guided waves in plate-like structures makes it possible to derive analytical solutions, by appealing to the first order Born approximation. The diffraction tomographic algorithm forms the basis for reconstructing the image of a structural damage, thus providing quantitative information on structural health.

INTRODUCTION

With the recent development in the field of smart materials and structures, structural health monitoring (SHM) is emerging as a promising technique to significantly transform aircraft maintenance and aircraft structural integrity management. A successful implementation of SHM will need to integrate two distinct strands that have traditionally been pursued separately, viz. damage detection and damage assessment. Here damage detection implies finding, in the order of increasing difficulty, the presence, location, and severity of potential damages. In this regard, considerable effort has been devoted to the development and application of Lamb-waves based techniques for damage detection in plate-like structures, which act as wave guides. Although the Rayleigh-Lamb solution is exact for plates, it is rather unwieldy for characterizing wave scattering by damages, due to the need to account for mode conversions among an infinite number of propagating and non-propagating wave modes. Consequently, a new approach is called for to quantitatively characterize the size and severity of damage.

The tomographic imaging method has recently been employed by Hinders et al [1, 2] to provide quantitative assessment of damage size, in which the straight-ray assumption was adopted, and thence the effect of wave scattering was not considered. The purpose of this paper is to present a plate-wave diffraction tomography approach that account for the effect of wave field diffraction. Mindlin plate theory is employed to describe the flexural (anti-symmetric) waves. The relative simplicity of Mindlin's plate theory as opposed to the exact Rayleigh-Lamb solution enables image reconstruction of a damage using a
diffraction tomography algorithm. This paper will focus on recently developed theoretical concepts and results for detecting delamination damage in fibre-composite, plate-like structures. The modeling is based on the premise that the primary effect of a delamination is to reduce locally the plate's flexural stiffness. Accordingly, for the purposes of detection, the delaminated region can be treated as an inhomogeneity, having a lower (and generally complex) flexural stiffness. The interaction of plate waves with such inhomogeneities is solved explicitly by appealing to the Born approximation, which is appropriate for detecting relatively low levels of damage (barely visible impact damage). The computational algorithm for implementing the diffraction tomography will be briefly discussed.

GREEN'S FUNCTION OF MINDLIN PLATE THEORY

Consider the scattering of an incident plate-wave by a localized inhomogeneity occupying a finite region $\Sigma$ bounded by a smooth contour $\partial\Sigma$, as shown in Fig. 1. The objective in this section is to derive an integral representation for the scattered field in terms of a distribution of induced sources over the region $\Sigma$. For this purpose, the governing equations for a plate of thickness $h$ and density $\rho$ can be expressed as follows,

$$M_{\rho\alpha,\beta} - Q_{\alpha} + \rho I \ddot{\Omega}_\alpha = m_\alpha,$$  \hspace{1cm} (1a)

$$Q_{\alpha,\alpha} - \rho h \ddot{w} = q,$$  \hspace{1cm} (1b)

where the parameters $M_{\rho\beta}$ denote the bending moments, $Q_{\alpha}$ the shear forces, $\Omega_{\alpha}$ the angle of rotations, $m_\alpha$ the distributed bending moments, and $q$ the distributed lateral pressure. The variable $I$ denotes the moment of inertia ($= h^3/12$). The usual conventions for derivatives ($\ddot{\Omega} \equiv \partial^2 \Omega/\partial t^2$, $\dddot{w} = \partial^3 w/\partial t^3$, and $t$ denotes time) and for summation over repeated subscripts apply. The bending moments $M_{\rho\beta}$ and the shear forces $Q_{\alpha}$ are related to the displacement fields,

$$M_{\rho\beta} = -D \gamma_{\rho\beta} = -D \left( \nu \ddot{\Omega}_{\alpha,\alpha} \delta_{\rho\beta} + \frac{1-\nu}{2} \left( \dddot{\Omega}_{\rho,\rho} + \dddot{\Omega}_{\beta,\beta} \right) \right),$$  \hspace{1cm} (2a)

$$Q_{\alpha} = \bar{\mu}h (w_{,\alpha} - \dddot{\Omega}_\alpha).$$  \hspace{1cm} (2b)

where $D$ denotes the bending stiffness ($= EI/(1-\nu^2)$), and $\bar{\mu}$ the effective shear modulus. Substituting (2a,b) into (1a,b) and including rotary inertia and transverse inertia terms, one obtains the following equations of motion for a homogeneous plate.

![FIGURE 1. Configuration for plate-wave scattering by an inhomogeneity (shadowed region).](image)
\[
D \left[ \nabla \nabla \cdot \mathbf{\Omega} - \frac{1 - \nu}{2} \nabla \times \nabla \times \mathbf{\Omega} \right] + \bar{\mu} h \left( \nabla w - \mathbf{\Omega} \right) - \rho I \ddot{\mathbf{\Omega}} = \mathbf{m}, \quad (3a)
\]

\[
\bar{\mu} h \nabla \cdot \left( \nabla w - \mathbf{\Omega} \right) - \rho h \ddot{w} = -q. \quad (3b)
\]

Although the rotation \( \mathbf{\Omega} \) is not strictly speaking a displacement, it is useful to regard the combination,

\[
\mathbf{u} = u_i = (-\Omega_x, -\Omega_y, w), \quad (4a)
\]
as defining a generalized displacement vector of plate theory, analogous to the elastic displacement \( \mathbf{u} \) of three-dimensional (3D) elasticity. In the same spirit, one can define a 3D vector of plate-theory traction \( \mathbf{T} \) acting at a point \( x = (x, y) \) on a curve \( C \) with (outward) normal \( \mathbf{n} \) by

\[
\mathbf{T} = T_i = (\mathbf{M}_a n_a, M_a n_a, \mathbf{Q}_a n_a), \quad (4b)
\]
and a body-force vector \( \mathbf{f} \) by

\[
\mathbf{f} = f_i = (m_x, m_y, q). \quad (4c)
\]

Fundamental to the solution of wave scattering are the dynamic Green’s functions \( g_{ij}(x, \xi) \), which represent the \( i \)-th component of the displacement vector \( \mathbf{u} \) at location \( x \) due to the \( j \)-th component of force \( f \) acting at an arbitrary point \( \xi \). The Fourier transform of the resulting displacements are mathematically related to the Fourier transform of the point forces,

\[
\begin{bmatrix}
\hat{\mathbf{u}}_x \\
\hat{\mathbf{u}}_y \\
\hat{\mathbf{w}} \\
\end{bmatrix} = \begin{bmatrix}
\hat{g}_{11} & \hat{g}_{12} & \hat{g}_{13} \\
\hat{g}_{21} & \hat{g}_{22} & \hat{g}_{23} \\
\hat{g}_{31} & \hat{g}_{32} & \hat{g}_{33} \\
\end{bmatrix} \begin{bmatrix}
\hat{m}_x \\
\hat{m}_y \\
\hat{q} \\
\end{bmatrix}. \quad (5)
\]

As an example, consider the case of point moment \( m(x, y) \), equation (3b) can be simplified into an axisymmetric form by setting,

\[
\hat{\phi} = \frac{\partial \hat{u}}{\partial y}, \quad \hat{\psi} = \frac{\partial \hat{v}}{\partial y}, \quad \hat{\psi} = -\frac{\partial \hat{\psi}}{\partial x}. \quad (6)
\]

The resulting equations can then be solved by using Hankel transform, leading to the following solutions for the three variables \( \hat{u}, \hat{v}, \) and \( \hat{\psi} \),

\[
\hat{u} = i \frac{\gamma H_0^{(1)}(k_1 r') - \gamma H_0^{(1)}(k_2 r')}{4D k_1^2 - k_2^2}, \quad (7a)
\]

\[
\hat{v} = i \frac{H_0^{(1)}(k_1 r') - H_0^{(1)}(k_2 r')}{4D k_1^2 - k_2^2}, \quad (7b)
\]

\[
\hat{\psi} = -\frac{H_0^{(1)}(k_1 r')}{2D(1 - \nu^2)k_1^2}, \quad (7c)
\]

where \( r' = |x - \xi| \). The Green functions \( \hat{g}_{11}, \hat{g}_{21}, \) and \( \hat{g}_{33} \) can now be readily derived from the above expressions. The other Green functions can be obtained by a similar method. The full expressions are given in [3].
SCATTERING BY WEAK INHOMOGENEITIES

Equations (1a,1b) hold equally for an inhomogeneous plate; however, the plate-theory parameters (i.e. the bending stiffness $D$, the shear stiffness $\mu_h = \tilde{\mu}h$, the rotary inertia $\rho_l = \rho h$, and the transverse inertia $\rho_h = \rho h$) now vary with position $x$. It is convenient to represent this variation in the following form,

$$
D = D[1 + s_1(x)], \quad \mu_h = \tilde{\mu}h[1 + s_2(x)], \quad \rho_l = \rho h[1 + s_3(x)], \quad \rho_h = \rho h[1 + s_4(x)]. \tag{8}
$$

The variations $s_n(x)$ ($n=1,2,3,4$) are non-zero for $x \in \Sigma$, but vanish for $x \notin \Sigma$. It is noted that these plate-theory parameters are not material properties: they depend on the thickness $h$, i.e. they depend on the geometrical configuration as well as the intrinsic material properties, unlike the case for inhomogeneities in 3D elasticity [4]. A variation in $D$ could result from a variation in the Young’s modulus, due for instance to microcracking, or it could result from a variation in thickness due to corrosion thinning, say, or it could be the result of variation in bending stiffness due to delamination, without a change in the overall plate thickness. From the viewpoint of a plate theory, these various possibilities cannot be distinguished on the basis of the variation $\partial D$ alone. However, a change in thickness $h$ would also cause a variation in the transverse inertia $\rho_h$, whereas a delamination would leave $\rho_h$ unchanged. Thus, one can anticipate that the identification of defects or damage from scattered-wave data will require a careful examination of the correlations between contributions attributable to variations in each of the four plate-theory parameters.

Substituting Eq.(8) into the expressions for the bending moment and shear force, Eqs.(2a,2b), one can express the equations of motion for a plate with an inhomogeneity in the same form as Eq.(2a,2b), except that the distributed moment and force are given by

$$
m_n = -\left\{ D(s_1 \Gamma_{s_n})_y + \mu_h s_2 \left( \omega_{\alpha} - \Omega_{\alpha} \right) - \rho h s_4 \Omega_{\alpha} \right\}, \tag{9a}
$$

$$
q = -\left\{ \mu h \nabla \cdot \left[ s_2 (\nabla \omega - \Omega) \right] - \rho h s_4 \omega \right\}. \tag{9b}
$$

Because the equations of motion are linear, the total displacement field $u(x,t)$ in Fig.1 can be expressed as a superposition of two terms,

$$
u = u^i + u^s, \tag{10a}
$$

corresponding to the incident wave and the scattered wave, designated by superscripts $I$ and $S$ respectively. The incident field $u^i$ is assumed to be specified, while the scattered field $u^s$ is to be determined from the governing equation, which can be written in the form,

$$
L[u^s] = \begin{cases} 
  f[u] & x \in \Sigma \\
  0 & \text{otherwise} 
\end{cases}, \tag{10b}
$$

where $L[\cdot]$ denotes the differential operator in Eqs.(2a,2b), $f[u] = (m_x, m_y, q)$ denotes the terms in Eqs.(9a, 9b), and the following condition has been used

$$
L[u^I] = 0, \tag{10c}
$$

$i.e.$ the sources of the incident field are assumed to be outside the domain of interest. Recalling the dynamic Green’s functions, the scattered field can be represented as follows,
\[ \hat{u}^S(x, \omega) = \int_{\Sigma} \mathcal{F}_j[u(\xi)] \hat{g}_y(x|\xi, \omega) d^2\xi, \]  

(11)

where \( \xi \) denotes an arbitrary point \((\zeta, \eta)\) within the source region \( \Sigma \), and \( d^2\xi = d\zeta d\eta \). This provides a first representation for the scattered field. It is not, however, the most convenient representation for detailed calculations, because the source distributions given by Eqs.(9a, 9b) involve spatial derivatives. A more convenient representation can be obtained through integration by parts, which leads to

\[ \hat{u}^S(x, \omega) = \int_{\Sigma} \left\{ D_{ij} \hat{F}_{\mu\nu} \hat{g}_{ij,\alpha,\beta} + \mu h s_2 (\hat{w}_{i\alpha} - \hat{\Omega}_{i\alpha} \hat{\gamma}_{i\alpha}) \hat{g}_{ij,\alpha,\beta} + \omega^2 \rho h s_3 \hat{\Omega}_{ij,\alpha} + \omega^3 \rho h s_4 \hat{g}_{ij,\beta} \right\} d^2\xi \]  

(12)

where the variations \( s_1(x) \) and \( s_2(x) \) are assumed to be non-uniform within the region \( \Sigma \), but vanish on the boundary \( \partial \Sigma \) \( [4-6] \) i.e.

\[ s_1(x, \omega) = s_2(x, \omega) = 0, \quad x \in \partial \Sigma. \]  

(13)

It can be seen that the scattered field depends on (i) the variations in the constitutive parameters \( (s_1, s_2, s_3, s_4) \) and (ii) the displacement field \( \hat{w}, \hat{\Omega}, \hat{F} \), where \( \hat{F} \) denotes the plate-theory strains defined by Eq.(2b).

As noted earlier, Eq.(12) constitutes an integral equation for the scattered field \( \hat{u}^S \), which, in general, requires a numerical solution. The objective in this section is to derive an explicit representation for \( \hat{u}^S \) for the case of a weak inhomogeneity, i.e.,

\[ \| s_n(x) \| = o(1), \quad n = 1, \cdots, 4. \]  

(14)

It then follows that the total displacement \( \hat{u} = \hat{u}' + \hat{u}^S \) is well approximated by \( \hat{u} = \hat{u}' \) on the right hand side of Eq.(12), one obtains the following approximate representation for \( \hat{u}^S \),

\[ \hat{u}^B(x, \omega) = \int_{\Sigma} \left\{ D_{ij} \hat{F}_{\mu\nu} \hat{g}_{ij,\alpha,\beta} + \mu h s_2 (\hat{w}_{i\alpha} - \hat{\Omega}_{i\alpha} \hat{\gamma}_{i\alpha}) \hat{g}_{ij,\alpha,\beta} + \omega^2 \rho h s_3 \hat{\Omega}_{ij,\alpha} + \omega^3 \rho h s_4 \hat{g}_{ij,\beta} \right\} d^2\xi \]  

(15)

where the superscript \( I \) identifies the incident wave. The superscript \( B \) is used to denote the (first order) Born approximation for the scattered field. This expression can be further simplified if one makes specific assumptions for (i) the incident field \( \hat{u}' \); (ii) the spatial variation of the parameter perturbations \( s_n (n = 1, \cdots, 4) \), as discussed in the following subsections.

**PLANE WAVE DIFFRACTION TOMOGRAPHY**

If \( \hat{u}' \) is assumed to be a plane wave, \( \hat{u}^B \) can be shown to reduce to a 2D spatial Fourier transform of the parameter perturbations \( s_n(x) \). To illustrate this result in the simplest case, consider \( \hat{u}' \) to represent a pure mode 1 plane wave, specified by,

\[ \hat{w}'(x, \omega) = e^{ik'_1 \cdot x}, \quad \hat{\Omega}' = i \gamma_1 k'_1 \hat{w}', \]  

(16)

where \( k'_1 = k_1 (\cos \theta', \sin \theta') \), as shown in Fig.2. The parameter \( \gamma_1 \) is given by [3],

\[ \gamma_1 = 1 - \rho \omega^2 / \mu h \]  

(17)

The Born approximation (15) can then be shown to reduce to a sum of four contributions, corresponding to the four parameter perturbations, as follows,
FIGURE 2. (a) Scattering of (mode 1) plane wave by an inhomogeneity for $\omega < \omega_c$. (b) The point in the transform-parameter plane corresponding to a fixed $k_1'$.

\[ \hat{w}^B(x, \omega) = \int_{\Sigma} \{ D_{\alpha i} \hat{\xi}_d \hat{g}_{3\alpha \beta} + \mu h_{\alpha} (\hat{w}^{-} - \hat{\Omega}_\alpha \hat{g}_{3\alpha} + \hat{g}_{33\alpha}) \\
+ \rho \omega^2 \bar{l}_{\beta} \hat{\xi}_d \hat{g}_{3\alpha} + \rho \omega^2 h_{\alpha} \hat{g}_{33} \} d^2 \xi \]  

(18)

Substituting (16) and replacing the Hankel function by the following plane-wave expansion

\[ H_0^{(1)}(k_1 | x - \xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i[k(x-\xi) + \eta(|y-\xi|)]} d\kappa \]  

(19)

where

\[ \eta = \sqrt{k_1^2 - \kappa^2} \]  

(20)

Upon simplification the following expression can be obtained for $y > \xi_y$, corresponding to forward scattering,

\[ \hat{w}^B(x, \omega) = \sum_{n=1}^{4} U_n(\omega) \int_{-\infty}^{\infty} f_n(\theta') \frac{d\kappa}{\eta} \int_{\Sigma} s_n(\xi) \exp[i\theta] \exp[i(\kappa(x-\xi) + \eta(|y-\xi|))] d^2 \xi \]  

(21a)

\[ U_1(\omega) = -\frac{i\gamma_1 k_1^2}{4\pi(k_1^2 - k_2^2)} \]  

(21b)

\[ U_2(\omega) = -\frac{i\mu h k_1}{4\pi D(k_1^2 - k_2^2)} \frac{1}{\gamma_1} \]  

(21c)

\[ U_3(\omega) = -\frac{i\rho \omega^2 \bar{l}_1 \gamma_1 k_1}{4\pi D(k_1^2 - k_2^2)} \]  

(21d)

\[ U_4(\omega) = -\frac{i\rho \omega^2 h}{4\pi D(k_1^2 - k_2^2)} \]  

(21e)

\[ f_1(\theta') = (\kappa \cos \theta' + \eta \sin \theta')^2 + \nu(\kappa \sin \theta' - \eta \cos \theta')^2 \]  

(21f)

\[ f_2(\theta') \equiv f_3(\theta') = \kappa \cos \theta' + \eta \sin \theta' \]  

(21g)

\[ f_4(\theta') = 1 \]  

(21h)
Now defining the two-dimensional Fourier transform of the source density as
\[ \hat{s}_n(K) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_n(\xi)e^{-iK\cdot \xi} d^2\xi \]  
(22)
Recognizing part of the inner integral (21a) as the two-dimensional Fourier transform of the source density evaluated at frequency \((\kappa - k_1 \cos \theta^I, \eta - k_1 \sin \theta^I)\) we find,
\[ \hat{w}^B(x,\omega) = \sum_{n=1}^{4} U_n(\omega) \int_{-\infty}^{\infty} f_n(\theta^I, \kappa) \hat{s}_n(\kappa - k_1 \cos \theta^I, \eta - k_1 \sin \theta^I)e^{i(\kappa x + \eta y)} \frac{d\kappa}{\eta} \]  
(23)
Taking the Fourier transform of the above expression over \(x\) along the receivers \((y = \ell)\), with the transform parameter \(\alpha\) and using the following property of Fourier integrals,
\[ \int_{-\infty}^{\infty} e^{i(x-\alpha)x} dx = 2\pi \delta(\kappa - \alpha) \]  
(24)
we find
\[ \int_{-\infty}^{\infty} \hat{w}^B(x,\ell)e^{-i\alpha x} dx = \frac{2\pi}{\eta} e^{i\theta^I} \sum_{n=1}^{4} U_n(\omega) f_n(\theta^I, \alpha) \hat{s}_n(\alpha - k_1 \cos \theta^I, \eta - k_1 \sin \theta^I) \]  
(25)
with
\[ \eta = \sqrt{k_1^2 - \alpha^2} \]  
(26)
The factors \(U_n(\omega) f_n(\theta^I, \alpha)e^{i\theta^I}\) are constant for a fixed receiver line \((y = \ell)\) and a known incident plane-wave \((\theta^I, k_1)\).

For a fixed \(\theta^I\), the equation (25a) provides information on the 2D Fourier transforms \(\hat{s}_n(K)\) at a point lying on a circle \(|K - k_1^I| = k_1\) in the Fourier transform-parameter plane, as shown in Fig.2(a), when the integration variable \(\alpha\) varies from \(-k_1\) to \(k_1\). By varying the angle \(\theta^I\) between 0 and \(\pi\), the semicircular arcs map out two disks centered at \((-k_1,0)\) and \((k_1,0)\), each of radius \(k_1\) as shown in Fig.3(a). This twin-circle spectral coverage is similar to that first reported by Devaney [7] in the context of well-to-well configuration of sources and receivers. Since the mapped region is symmetrical with respect to the origin, no further advantage can be gained from real valued source density.

If back-scatter response, i.e., values of \(\hat{w}\) along the lower row of receivers \(y = -\ell\), are also used for reconstruction, the values of \(\hat{s}_n(K)\) are also known for points on the semi-circle ADC of Fig.2(b). Consequently the Fourier transform is known on a complete circle. By varying the angle \(\theta^I\) between 0 and \(\pi\), the circles map out a heart-shaped region as shown in Fig.3(b). Unlike the previous case, the mapped region is not symmetrical with respect to the origin. Therefore, for real-valued source density, which leads
\[ \hat{s}_n^* (K, \omega) = \hat{s}_n (-K, \omega) \]  
(27)
advantage can be gained by making use of the complex conjugate of the transformed values over the reflected region about the origin. Consequently, the Fourier transforms of the source density are known over a disk centered at the origin of a radius \(2k_1\).
If surface average strain $\nabla \cdot \hat{\omega}$ is measured experimentally, instead of the deflection $\hat{w}$, it can readily be shown that

$$\nabla^{2} \hat{\omega}(x, \omega) = -\gamma_{1}k_{1}^{2} \hat{w}^{\beta}(x, \omega),$$

since the Green’s functions satisfy the following relationships,

$$g_{\alpha, \beta} = -\gamma_{1} \nabla^{2} g_{3\alpha}, \quad \alpha = 1, 2, 3.$$  (29)

The sum of surface strains can be most readily measured by using circular piezoelectric sensors. Consequently, the measurement of piezoelectric sensors can be directly employed for image reconstruction.

CONCLUSIONS

Solutions have been obtained for forward and inverse scattering of plate-waves by weak inhomogeneities, based on first order Born approximation. The reconstruction formula for plane-wave, generated by phase-arrays of actuator and receiver pairs, has been developed.

REFERENCES