AN EXTENSION OF THE SPRING MODEL TO NONLINEAR INTERFACES

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ABSTRACT. This paper presents a new set of boundary conditions to be enforced on an elastodynamic wave interacting with a nonlinear interface. They are formulated within the framework of the quasi-static approximation, and model the effect of defects on the interface properties by means of two distributions of nonlinear springs. The nonlinear terms are derived from the micromechanics of two rough surfaces in contact. Numerical results are presented both for normal and oblique incidence.

INTRODUCTION

Stress-corrosion cracks result of the combined action of chemical agents present in the harsh environment in which the component operates, and stress fields of thermal origin or due to external loads. The geometry of stress-corrosion cracks can be rather complex, with several branches and surfaces that are rough both at the scale of the grains' typical dimension and in the sub-micron region. Because of the random nature of the corrosion process, the surfaces of such cracks do not conform to each other at the scale that characterizes the range of action of the chemical forces.

Stress-corrosion cracks constitute a serious threat to the safety of nuclear power plants. Early detection of such defects is, therefore, of paramount importance. To allow the assessment of its structural integrity, a nuclear power plant must be temporarily shut down. Under these conditions, the stress fields that are responsible for the growth of the crack disappear, and partial closure of the defect may take place. A crack that is partially closed can be difficult to detect by conventional methods since it tends to become transparent to an inspecting beam of ultrasonic waves. To further worsen the problem, the microstructure of the hosting material, by linearly scattering part of the energy of the inspecting beam, may raise the signal threshold above which the defect signal can be detected.

In the recent past, considerable evidence has been gathered in laboratory experiments to show that cracks and imperfect interfaces can behave in a nonlinear fashion [1-10]. In particular, they may act as sources of elastodynamic waves with spectral properties that are distinct from those of the inspecting beam. Hence, it is
reasonable to conceive that phenomena such as higher harmonic generation and frequency mixing may be exploited for the purpose of detecting partially closed cracks that otherwise would not be sensed by linear, conventional methods.

In the attempt to contribute to the development of novel inspection methods, this work focuses on modelling the nonlinear interaction of a bulk plane wave with a nominally flat interface formed by two rough surfaces in contact. The goal of this investigation is the formulation of suitable boundary conditions, which may be enforced at the surfaces of a partially closed crack as well as at the surface of a kissing bond. Thus, in the next section, the boundary conditions that extend those of the well-known spring model [11, 12] to the nonlinear case will be presented. A detailed description of the relation between the interface macroscopic parameters and the micromechanics of the interface is deferred until a later publication. Here, only the most important features of the macroscopic elastic contacts of the interface will be discussed. Numerical results showing the nonlinear response of two contacting surfaces to an inspecting plane wave at both normal and oblique incidence will be presented, and their potential relevance to the nondestructive detection of partially closed cracks discussed.

THEORY

In the long wavelength limit, the scattering of an ultrasonic wave from an imperfect interface can be described by modelling the effect of distributed imperfections with two distributions of springs having normal and transverse elastic constants $K_N$ and $K_T$, respectively. For the specific case of two rough surfaces in contact, expressions for $K_N$ and $K_T$ - in terms of geometrical parameters of the rough surfaces and physical properties of the material - can be readily derived from the interface constitutive relations. For a recent review of the literature on this subject, and the details of the derivation, see reference [12].

Figure 1 illustrates the nonlinear dependence of the normalized interfacial elastic constants on the normalized relative approach, $\delta/Z_0$. Here, $Z_0$ is the distance between the mean planes of the two surfaces when there is no external load acting on them. The relative approach, $\delta$, measures the variation of the distance between the mean planes of the rough surfaces, which occurs under the action of an externally applied load. The profiles of the two surfaces are assumed to be statistically identical. The stiffness constants are normalized with respect to the product $Z_T \omega$, where $Z_T$ is the transverse

![Normalized interfacial stiffness constants versus normalized relative approach. These plots have been obtained for the steel-steel interface # 2 of Table 1.](image-url)
TABLE 1. Statistical parameters of the interfaces (adapted from reference [11]).

<table>
<thead>
<tr>
<th>Interface 1</th>
<th>Interface 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roughness (μm)</td>
<td>0.68</td>
</tr>
<tr>
<td>M (GPa/(μm^3))</td>
<td>5.4</td>
</tr>
<tr>
<td>Degrees of Freedom</td>
<td>3</td>
</tr>
</tbody>
</table>

acoustic impedance of the material, and ω is the angular frequency of the incident wave. To account for the most relevant nonlinear elastic properties of the interface, the first order approximation of $K_N$ and $K_T$ are considered, which are valid for small but not negligible values of $\delta$. In addition, since the transverse stiffness constant also depends on the tangential extra-displacement $\Delta u_t$, the following expansions are considered for $K_N$ and $K_T$, respectively

$$K_N(\delta + \Delta \delta) = K_{N,0} + \frac{dK_N}{d\delta} \Delta \delta = K_{N,0} - K_{N,\delta} \Delta u_t,$$  \hspace{1cm} (1)

$$K_T(\delta + \Delta \delta, \Delta u_t) = K_{T,0} + \frac{\partial K_T}{\partial \delta} \Delta \delta + F(\Delta u_t).$$  \hspace{1cm} (2)

In Equations (1) and (2), the identity $\Delta \delta = -\Delta u_t$ has been used, where $\Delta u_t$ is the normal component of the interface’s extra-displacement. While the evaluation of the derivatives with respect to $\delta$ is straightforward once the pressure vs. $\delta$ relation is considered (see [12, 13], for example), the evaluation of the dependence of $K_T$, which is valid for small but not negligible $\Delta u_t$, is more complex. In fact, as Mindlin and Deresiewicz [14] showed, the force-displacement relation for two spheres in contact and subjected to an oscillating tangential force is hysteretic, and leads to a discontinuous dependence of the tangential stiffness on $\Delta u_t$. For small values of the latter, the following expression for $K_T$ can be derived, which is valid on both paths of the hysteretic loop

$$K_T = K_{T,0} - \frac{1}{2} K_{T,\delta} \left[ \text{sgn} \left( \frac{\partial \Delta u_t}{\partial t} \right) \Delta u_t + \Delta u_t^\text{max} \right].$$  \hspace{1cm} (3)

In Equations (3), $t$ represents time, and $\Delta u_t^\text{max}$ is the maximum absolute value reached by $\Delta u_t$ during each loop. The function $\text{sgn}(\cdot)$ is equal to 1 when its argument is positive, and to -1 when the latter is negative. Note that at the beginning of each half-cycle, the stiffness is equal to its 'linear' value $K_{T,0}$. Equation (2) then becomes

$$K_T(\Delta u_t, \delta - \Delta u_t) = K_{T,0} - K_{T,\delta} \Delta u_t - \frac{1}{2} K_{T,\delta} \left[ \text{sgn} \left( \frac{\partial \Delta u_t}{\partial t} \right) \Delta u_t + \Delta u_t^\text{max} \right]$$  \hspace{1cm} (2')

where $K_{T,\delta} = \partial K_T/\partial \delta$, and $\Delta \delta = -\Delta u_t$ has been used.

Within the framework of the distributed spring model, the boundary conditions describing the nonlinear interaction of an elastodynamic wave with an imperfect
interface are therefore

\[ \sigma_{3,1}^1(0^+) = (K_{N,0} - K_{N,1} \Delta u_1) \Delta u_1, \]  

\[ \sigma_{3,1}^0(0^+) = K_{T,0} \Delta u_1 - \frac{1}{2} K_{T,1} \left[ \text{sgn} \left( \frac{\partial \Delta u_1}{\partial t} \right) \left( \Delta u_1^2 - \Delta u_{1,\text{max}}^2 \right) - \Delta u_{1,\text{max}} \Delta u_1 \right] - K_{T,N} \Delta u_1 \Delta u_1, \]  

\[ \sigma_{3,1}^0(0^+) = \sigma_{3,1}^1(0^+), \]  

\[ \sigma_{3,1}^0(0^+) = \sigma_{3,1}^0(0^+). \]  

To solve Equations (4a-d), the field variables and the interface elastic constants are normalized as follows,

\[ u_i = A_{in} U_i, \quad i = 1, 3; \quad x_i = k_T X_i, \quad i = 1, 3; \quad \omega t = \tau, \]

\[ K_{a,0} = K_{a,0}/Z_T, \quad \omega, \quad \varepsilon_{a,1} = \frac{K_{a,1}}{K_{a,0}} A_{in}, \quad \alpha = N, T, \]

where \( A_{in} \) is the amplitude of the incident wave, and \( k_T = \omega/C_T \) is the shear wavenumber. It can be shown that, after normalization, the nonlinear parameter \( K_{TN} \) leads to \( \varepsilon_{N,1} \), reducing the number of independent nonlinear parameters to two. Figure 2 shows the dependence of the two nonlinear parameters \( \varepsilon_{N,1} \) and \( \varepsilon_{T,1} \) on the normalized normal stiffness, \( K_{N,0} \), for the two interfaces of Table 1. These curves have been obtained for values of \( A_{in} = 3 \text{ nm} \), and \( \omega/2\pi = 1 \text{ MHz} \). With the exception of a small neighbourhood of \( K_{N,0} = 0 \), in which the contacts are not fully developed, the normalized parameters are always much smaller than unity. A perturbation approach

![Figure 2](image-url)
seems, therefore, the obvious method to search for the solutions of the problem. It also suggests the form of the solutions:

\[ U_i^{+\pm} = U_i^{\pm} e^{i\tau} + (e_{N, i} V_i^{+\pm} + e_{T, i} W_i^{+\pm}) e^{j\tau} + \ldots, \quad i = 1, 3, \quad (5) \]

in which the superscripts '+' and '-' refer to the fields in the upper and lower half-spaces, respectively. In the following, particular solutions of the normalized set of equations will be presented.

**NUMERICAL RESULTS**

**Normal Longitudinal Incidence**

The problem at hand requires the use of Equations (4a) and (4c) only. Here and in the following sections, the frequency, \( \omega / 2 \pi \), and the amplitude, \( A_m \), of the incident wave are equal to 1 MHz, and 3 nm, respectively. The solutions represent waves that are longitudinally polarized, and to the first order approximation in the \( e_{N, 1} \) are of the following form,

**Order 0:** \[ U_0^R = R = \frac{1}{1 - j 2 \bar{K}_{N, 0}/\kappa}, \quad U_0^T = T = \frac{-j 2 \bar{K}_{N, 0}/\kappa}{1 - j 2 \bar{K}_{N, 0}/\kappa}, \quad (6) \]

where \( R \) and \( T \) are the reflection and transmission coefficients at normal incidence, and \( \kappa \) is the ratio between the longitudinal and shear phase velocities; and

**Order 1:** \[ V_i^- = V_i^+ = \frac{F}{2 \bar{K}_{N, 0}/\kappa}, \quad (7) \]

where \( F = \frac{\bar{K}_{N, 0}}{2 \kappa} \left[ \text{Re}(T - 1 + R) + j \text{Im}(T - 1 + R) \right]^2 \). \quad (8)

The time dependence of the first-order corrections is \( e^{j2\tau} \). Figure 3a illustrates the dependence of the first-order solutions \( e_{N, 1} V_i^{+\pm} \) on \( \bar{K}_{N, 0} \) for the two interfaces of Table 1. The interface with the smaller roughness shows a more pronounced nonlinear response, which in both cases reaches its maximum near \( \bar{K}_{N, 0} = 1 \).

**Normal Shear Incidence**

Equations (4b) and (4d) are used to solve this problem. Following the same procedure as in the previous case, the solutions of the zero-th order system are recovered and found to be identical to those in Equation (6), except for the absence of the factor \( \kappa^{-1} \) multiplying \( \bar{K}_{T, 0} \). As for the first order correction, its spectrum is found to contain a cascade of odd harmonics, similarly to the case of an interface with boundary conditions governed by friction's law which was considered by O’Neil *et al.*
FIGURE 3. First-order corrections vs. normalized stiffness for normal longitudinal (a) and shear (b) incidence. The two interfaces are those of Table 1. The amplitude of the incident wave is used to normalize that of the higher harmonics.

[10]. This feature is a consequence of the 'source' term being an odd function of time. The 'source' function is the right-hand side of Equation (4b) for the first-order system, and is determined by the solutions of the 0-th order system. The partial components of the first-order correction can be found by Fourier transforming the 'source' function, and solving the corresponding system of linear equations. The complex amplitude of the m-th Fourier component of the first-order correction (apart from the factor $\epsilon_T$) is found to be

$$W_{m}^{+,-} = -j \frac{2K_T}{m-j2K_T} C_m, \quad m = 1, 3, 5, \ldots$$ (9)

where $C_m$ is the m-th Fourier complex coefficient of the 'source' function. Figure 3b presents the graphs of the first three odd harmonics versus the normalized transverse stiffness for the second interface (Interface 2) of Table 1. As with the corrections considered in the case of longitudinal incidence, each component reaches its maximum around $K_{T,0}=1$ to decrease rapidly soon after. A comparison between the amplitudes of the Fourier components of Figures 3a and 3b shows that the nonlinear response to a longitudinal incident wave is more than 20 dB higher than that to a shear wave. This difference can be explained in part by the higher-order (third and higher) harmonic content that characterizes the latter.

**Oblique Incidence**

The case of a longitudinal wave at oblique incidence can be easily solved by means of the same method. The nonlinear contributions to the first order scattered waves contain second harmonic terms that are proportional to $\epsilon_{N,1}$, and odd higher-order harmonics proportional to $\epsilon_{T,1}$. Of the two, the second harmonic is again more pronounced, reaching values at normal incidence that approach $-35$ dB below the
fundamental one for values of the normalized interfacial stiffness $K_{N,0}$ near unity. The amplitude of the third order harmonic is negligible for angle of incidence smaller than 15 degrees, and approaches about −50 dB at angle of incidence around 75 degrees.

Of greater interest is the case of oblique shear incidence. Figure 4 shows the reflection (a) and transmission (b) coefficients, which are two of the solutions of the zero-th order system, for a shear incident wave that is mode-converted into a longitudinal wave. The feature of interest is the large peak of the reflection coefficient of longitudinal wave at the longitudinal critical angle, $\theta_L = 34^\circ$. The large energy density build-up in the region proximal to the interface for values of the angle of incidence around $\theta_L$ is responsible for the considerable increase of the nonlinear response, especially in the form of longitudinal second harmonic (Fig. (5)). Note that for values of the normalized normal stiffness proximal to one the amplitude of the second harmonic can be only one order of magnitude lower than that of the fundamental. Also worth notice is the magnitude of the shear second harmonic that is generated at angles of incidence equal to or larger than $\theta_L$.

![FIGURE 4. Longitudinal reflection (a) and transmission (b) coefficients of a shear incident wave for various values of the normalized normal interfacial stiffness as functions of the angle of incidence.](image)

![FIGURE 5. Longitudinal (a) and shear (b) second harmonic radiated by a nonlinear interface insonified by a shear wave as a function of the angle of incidence.](image)
SUMMARY

A set of nonlinear boundary conditions to be enforced at the interface formed by two rough surfaces in contact has been proposed. The interfacial parameters have been derived from well-known models based on the interaction between asperities that can be approximated by spheres in the contact region. The nonlinearity associated with the normal interfacial response is shown to be of classical type, while that associated to the transverse response is of hysteretic type. The interfacial nonlinearities have been shown to be much smaller than the linear effects, so that a perturbation approach to solving the nonlinear reflection and transmission problems applies naturally. The boundary value problem has been solved for arbitrary wave polarization and angle of incidence. For a longitudinal wave at normal incidence, the leading correction to the linear response is shown to be proportional to the square of the interfacial normal displacement discontinuity and to contain both a constant term and the second harmonic. For a shear wave at normal incidence, the nonlinear response consists of a cascade of odd harmonics. Of the two, the former dominates, as could be expected from simple considerations based on the dependence of the coefficients of the series on their order. At oblique incidence the most interesting result is found in the region around the longitudinal critical angle for an incident shear wave. As the amplitude of the scattered longitudinal wave reaches a maximum at the longitudinal critical angle, a large part of the incident energy is confined within a narrow region around the interface. Such a high energy density causes an enhanced nonlinear response of the interface. Before the critical angle is reached, the latter consists of a second harmonic longitudinal wave at grazing incidence. This fact may be of relevance to improving the detection of a partially closed crack, and, possibly, to identifying its location by nonlinear methods.

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REFERENCES