SCATTERING OF ELASTIC WAVES BY A RECTANGULAR CRACK
IN A THICK-WALLED ANISOTROPIC SOLID

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ABSTRACT. The scattering of elastic waves by a rectangular crack in a half-space of arbitrary anisotropy is considered. The scattering problem is formulated as a hypersingular integral equation for the crack-opening-displacement by means of the half-space Green’s tensor, which is solved by expanding in a double series of Chebyshev functions. Models of transmitting and receiving transducers give a complete model of ultrasonic testing. A few numerical examples are given.

INTRODUCTION

The modeling of ultrasonic nondestructive testing is useful for a number of reasons. It enhances the physical understanding of the wave propagation which is of particular relevance in anisotropic media. Modeling is also very useful for parametric studies, development of testing procedures and the qualification of procedures and personnel.

In isotropic media the scattering by cracks is an old subject with many references, see Martin and Wickham [1] and Zhang and Gross [2] for surveys. Rectangular cracks are considered by Itou [3] and Guan and Norris [4] and a rectangular crack near a free planar surface is treated by Jansson [5]. In anisotropic media there is less work done, see Zhang and Gross [2] and Zhang [6] for surveys. Especially in 3D there seems to be very few contributions, but Mattsson et al. [7] have considered a strip-like crack. Here a rectangular crack in an anisotropic component is considered and the crack may, furthermore, be located close to a planar back surface. Models of ultrasonic transducers in both transmission and reception are included and a complete model of an ultrasonic NDT situation is thus obtained.

PROBLEM FORMULATION

The scattering geometry of interest is depicted in Fig. 1. A rectangular crack with sides $2a_1$ and $2a_2$ is located in a thick-walled anisotropic solid. On the scanning surface two ultrasonic transducers, one transmitter and one receiver, are located. It is assumed that the distance between the crack and the scanning surface is large enough (at least a couple of wavelengths) so that multiple scattering between them can be neglected. However, the multiple scattering between the crack and the back surface is taken care of, so the distance between them can be arbitrary. The purpose here is to compute the change of the signal in
Figure 1. The modeling geometry with the rectangular crack, the back surface, the two transducers on the scanning surface, and the five different coordinate systems.

the receiver due to the presence of the crack. Possible contributions that are directly reflected from the back surface are not included.

The material of the thick-walled component is assumed to be linearly elastic and of general anisotropy. The density is denoted by \( \rho \) and the elastic constants by the fourth rank tensor \( \epsilon_{jmn'\nu} \). Due to the symmetries of stress and strain and the existence of a strain energy function, \( \epsilon_{jmn'\nu} \) has at most 21 independent components. Often the materials used in practise have a higher degree of symmetry, such as transversely isotropic materials, with 5 independent components, or orthotropic materials, with 9 independent components.

In Fig. 1, five different coordinate systems are introduced with locations and orientations as indicated. The system \( x^a \) is the crystal axis system of the anisotropic material. The vector from the origin of the transmitter system to the origin of the crack system is \( d \) and the vector from the origin of the receiver system to the origin of the crack system is \( r \). The distance between the back surface and the center of the crack is \( b \) (in the \( x^3 \) direction). The superscripts 't', 'r', 'c', 'b', and 'a' on quantities indicate in which coordinate system they are represented. The usual transformation rules for tensors apply with the transformation (rotation) matrices \( R^c \) (from \( x^b \) to \( x^c \)), \( R^b \) (from \( x^a \) to \( x^b \)), and \( R^a \) (from \( x^c \) to \( x^a \)). The rotation matrices can be parametrised by Euler angles, for instance.

Time harmonic conditions are assumed and the factor \( \exp(-i\omega t) \) is suppressed throughout. Here \( \omega \) is circular frequency and \( t \) is time. To solve the scattering problem an integral equation method is employed. As the multiple scattering between the crack and scanning surface is neglected, the scattering by a rectangular crack in an anisotropic half-space is to be solved. Starting from an integral representation, an integral equation for the crack-opening-displacement (COD) is obtained if the traction operator is applied, the field point is approaching the crack, and the boundary condition is used:

\[
\lim_{y^3 \to 0^+} \int_{-\alpha_1}^{\alpha_2} \int_{-\alpha_2}^{\alpha_2} \Delta u^c_j(x_1^c, x_2^c) S^c_{j\nu}(x_1^c, x_2^c, 0; y_1^c, y_2^c, y_3^c) \, dx_1^c \, dx_2^c = -\sigma_{ab}^{inc}(y_1^c, y_2^c, 0). 
\]

Here, \( \Delta u^c_j \) is the COD and \( \sigma_{ab}^{inc} \) is the traction vector due to the incident field on the crack. Note that the incident field is required to satisfy the boundary condition on the back surface, otherwise an extra term would appear in the integral equation. The kernel in the integral equa-
tion, $S'_{jp}$, is obtained by operating twice with the traction operator (once on each argument) on the Green’s tensor, i.e.,

$$S'_{jp}(x'; y') = \delta_{jmn} \delta_{x'_m} \delta_{y'_n} \partial_{x'_m} G'_{mm}(x'; y').$$

(2)

The Green’s tensor here is the half-space Green’s tensor. The integral equation (1) is of the hypersingular type, see Martin and Rizzo [8] for a discussion. Thus the limit in front of the integrations cannot be moved into the integrand. At a later stage the limit can be performed.

**SOLUTION OF THE INTEGRAL EQUATION**

The half-space Green’s tensor which is needed in the kernel of the integral equation (2) is decomposed into a free space and a back surface part. Both parts of the Green’s tensor are expanded in double Fourier transforms as

$$G_{ff}(x; y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_n^\pm U_{nj}^\pm U_{nj}^\pm e^{ik_n^\pm (x-y)} \, dk_1 \, dk_2, \quad x_3 > y_3,$$

(3a)

$$G_{fb,bb}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_n^b B_{nm} U_{mj}^b e^{ik_n^b (x-y)} \, dk_1 \, dk_2.$$

(3b)

In Eq. (3a) $U_{nj}^\pm = U_n^\pm (k_1, k_2)$ is component $j$ of the $n$:th displacement polarization vector corresponding to a wave propagating in the positive/negative $x_3$ direction. The associated wave vector is denoted by $k_n^\pm = (k_1, k_2, k_n^\pm (k_1, k_2))$. The corresponding traction polarization vector is denoted by $T_{nj}^\pm = T_n^\pm (k_1, k_2)$. The wave vectors and polarization vectors are obtained by solving a generalized eigenvalue problem, which is the result if a plane wave is inserted into the equations of motion and Hooke’s law. The integration constants $D_n^\pm = D_n^\pm (k_1, k_2)$ are determined by the jump conditions that the free space part of the Green’s tensor must satisfy. Finally, the reflection matrix $B_{nm'}$ is determined by the back surface boundary condition.

The kernel of the integral equation may now be calculated. First the Green’s tensor is expressed in the crack coordinate system $x^c$ and then the double traction operator (Eq. (2)) is applied to yield:

$$S'_{jp}(x^c; y^c) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} D_n^c T_{nj}^c T_{nj}^c e^{ik_n^c (x^c-y^c)}$$

$$+ \sum_{n'=1}^{3} D_n^{b, c} T_{j, j'}^{b, c} \left( B_{nn'} \right)^{b, c} \left( U_n^{b, c} \right)^{b, c} \left( U_{n'}^{b, c} \right)^{b, c}$$

$$\times e^{i(k_n^b c - k_n^b c) (x^c-y^c) + i(k_n^b c - k_n^b c)} \, dk_1 \, dk_2, \quad x_3^c > y_3^c.$$

(4)

In the above, the notation $\{\cdot\}^c$ stands for a rotation to the crack coordinate system. The notation $T_{j, j'}^{b, c}$ is used for the traction on the crack computed from $\{k_n^b\}^c$ and $\{U_n^{b, c}\}^c$, i.e., the wave and polarization vectors are first transformed to the crack system, then the traction on the crack is calculated by means of Hooke’s law.

In order to solve the integral equation (1) with the kernel (4), the COD is expanded in a suitable set of functions. Due to the simple geometry of the crack and the type of singularity along its edges, a convenient set of expansion functions is the Chebyshev functions given by

$$\psi_p(s) = \begin{cases} 
(1/\pi) \cos(p \arcsin s), & p = 1, 3, \ldots , \\
(i/\pi) \sin(p \arcsin s), & p = 2, 4, \ldots . 
\end{cases}$$

(5)
These functions have the form of a polynomial of order \( p - 1 \) times the factor \( \sqrt{1 - s^2} \) and thus contain the correct edge behavior at \( s = \pm 1 \). It should be noted that the behavior at the corners of the crack is expected to be different and does not seem to be known in the anisotropic case. No measure is therefore taken to include this behavior.

Using the set of functions (5), the expansion of the COD is

\[
\Delta u_j^q(x_1^t, x_2^t) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \beta_{pjq} \psi_p(x_1^t/a_1) \psi_q(x_2^t/a_2).
\] (6)

The expansion is inserted into Eq. (1), the equation is multiplied by \( \psi_{p'}(x_1^t/a_1) \psi_{q'}(x_2^t/a_2) \) and integrated over the crack. The result is the linear system of equations

\[
\sum_{j=1}^{3} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} Q_{j}^{p' q'} \beta_{pjq} = M_{j}^{p' q'}, \quad j' = 1, 2, 3, \quad p', q' = 1, \ldots, \infty.
\] (7)

In Eq. (7), the matrix \( Q_{j}^{p' q'} \beta_{pjq} \) is given by

\[
Q_{j}^{p' q'} \beta_{pjq} = (-1)^{p+q} \rho_j^{p' q'} \rho_j^{p' q'} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n=1}^{3} \left( D_n^{-1} T_{n j}^{-1} T_{n j}^{+1} \frac{J_p(k_{11}) J_p(k_{12}) J_q(k_{21}) J_q(k_{22})}{k_1^2 k_2^2} \right) + \sum_{n'=1}^{3} D_n^{n'} T_{n j}^{+1} \left( \frac{\{k_{n'}^{-1}\}^c}{\{k_{n'}^{+1}\}^c} \right) B_{n m'} T_{j}^{+1} \left( \frac{\{k_{m}^{-1}\}^c}{\{k_{m}^{+1}\}^c} \right) e^{i(k_{n'}^+ - k_{n}^-)^b} J_p \left( \frac{\{k_{n'}^{-1}\}^c a_1}{\{k_{n'}^{+1}\}^c a_1} \right) J_p \left( \frac{\{k_{n}^{-1}\}^c a_2}{\{k_{n}^{+1}\}^c a_2} \right) J_q \left( \frac{\{k_{n'}^{-1}\}^c a_2}{\{k_{n'}^{+1}\}^c a_2} \right)
\] (8)

and the right hand side is given by

\[
M_{j}^{p' q'} = \int_{-\infty}^{a_1} \int_{-a_2}^{a_2} \sigma_{3 j}^{c \text{in}}(y_1, y_2, 0) \psi_{p'}(y_1/a_1) \psi_{q'}(y_2/a_2) dy_1 dy_2.
\] (9)

The traction on the crack due to the incident field, \( \sigma_{3 j}^{c \text{in}}(y_1, y_2, 0) \), is specified in the next section.

**THE TRANSDUCERS**

Both the transmitting and receiving transducers are assumed to be of the conventional contact type. This can be modeled by prescribing the traction on the effective contact area of the transducer, see Niklasson [9] for further details. The traction modeling the transmitter is applied to a half-space made of the same material as the component containing the crack. The solution to this wave propagation problem is obtained in the same way as the free space Green’s tensor. The solution is

\[
u_j^{t \text{in}}(x^t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n=1}^{3} \xi_n^t T_{n j}^{t \text{in}} e^{i(k_{n}^- - x^t) \cdot k_{n}^-} d\xi_{n1}^t d\xi_{n2}^t,
\] (10)

where the wave and polarization vectors are computed in the transmitter system. Note the superscript 't-' since the positive \( x_2^t \) axis is pointing out of the component. The coefficients \( \xi_n^t = \xi_n^t(k_1^t, k_2^t) \) are the amplitudes of the plane waves in the spectrum and are determined by the boundary condition on the scanning surface modeling the transmitter.
In order to take the back surface into account, a reflection part is added to the incident field (as was done for the Green's tensor). Note that multiple reflections, i.e. waves reflected by the scanning surface, are not taken into account. After adding the reflection part and transforming the field to the crack system, the final result is

\[
\psi_j^{in}(x^c) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n=1}^{3} \sum_{n'=1}^{3} \left( \left( \mathbf{U}_{n''} \right) \right)_j e^{i\mathbf{k}_{n''}^c \cdot x^c} + \sum_{n'=1}^{3} \mathbf{B}_{n'n''} \left\{ \left( \mathbf{U}_{n''} \right) \right\}_j e^{i(\mathbf{k}_{n''}^c \cdot x^c)}
\]

The reflection matrix \( \mathbf{B}_{n'n''} \) is determined by the traction free boundary condition on the back surface.

Calculation of the traction vector on the crack from Eq. (11) and insertion into the expression for \( M_{p'q'} \), Eq. (9), yields

\[
M_{p'q'} = (-1)^{p+q} p'q' \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n=1}^{3} \sum_{n'=1}^{3} \left( \mathbf{J}_p \left( \left( \mathbf{k}_{n''}^c \right) \right)_j \mathbf{J}_q \left( \left( \mathbf{k}_{n''}^c \right) \right)_j \right) e^{i(\mathbf{k}_{n''}^c \cdot x^c)}
\]

Finally, in order to compute the output signal during an ultrasonic test, the receiver must be modeled. Here, no losses are included and the only coupling between the electrical and mechanical fields is piezoelectric. Therefore, the electromechanical reciprocity relation by Auld [10] is used. The reciprocity relation states that the change in electrical signal due to a defect may be computed by considering two states: (1) the transmitter is transmitting and the defect is present, and (2) the receiver is transmitting and the defect is absent. Auld’s reciprocity relation, in this case, may then be written as

\[
\delta \Gamma = -\frac{i\omega}{4P} \int_{-a_1}^{a_2} \int_{-a_3}^{a_3} \Delta \psi_j(x_1^c, x_2^c) \sigma^{re}_{Sj} (x_1^r, x_2^r, 0) \, dx_1^c \, dx_2^c,
\]

where \( \sigma^{re}_{Sj} \) is the traction when the receiving transducer is transmitting (in the absence of the crack). The quantity \( P \) appearing in Eq. (13) is essentially the power fed to the two transducers in the two states. The expression for the traction from the receiving transducer is identical to the expression for the traction from the transmitting transducer except that \( d \) is replaced by \( r \) and the superscripts 't' are replaced by 'r' throughout. Insertion into Eq. (13) results in the expression

\[
\delta \Gamma = -\frac{i\omega}{4P} \sum_{j=1}^{3} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \beta_{p'q'} M_{p'q'}^{re},
\]

where \( M_{p'q'}^{re} \) is computed from Eq. (12) with \( d \) replaced by \( r \) and the superscript 't' replaced by superscript 'r'. \( \beta_{p'q'} \) is the solution to the system of equations (7) and thus involves the transmitting transducer and the crack. It should be noted that the expression for \( \delta \Gamma \) above
holds at fixed frequency and is only the change in signal due to the crack. Also, as mentioned above, the multiple reflections via the scanning surface are not taken into account.

**NUMERICAL EXAMPLES**

When making numerical computations there are a few issues that must be addressed. The double integrals that appear in Eq. (8) and Eq. (12) must be computed with great care due to the cuts that appear in the wave numbers and due to the slow decay at infinity in Eq. (8) and the fast oscillations in Eq. (12).

In the examples the material of the component is chosen as a transversely isotropic austenitic steel (a weld material). This material has density 8120 kg/m³ and stiffness constants (in abbreviated notation with the 3 axis as symmetry axis) $C_{11} = 262.7$ GPa, $C_{33} = 216.0$ GPa, $C_{44} = 129.0$ GPa, $C_{12} = 98.2$ GPa, and $C_{13} = 145.0$ GPa. The material is only moderately anisotropic, but the slowness surface contains concave parts so for certain wave number ranges there exist two quasi SV waves.

Only a single transducer operating in pulse-echo mode is used. The transducer is square with side 10 mm. It is operating at the frequency 1 MHz and as only C-scans are computed only a single frequency is used as this gives a fair accuracy. The transducer is either a 0° P or a 60° SV transducer, where the angles are those resulting on an isotropic steel component. The results are left uncalibrated so it is only meaningful to compare results from the same transducer.

Figure 2 shows C-scans along a line from the 0° P probe for a horizontal square crack with side 8 mm and without back surface for four different tilts of the anisotropy. It is seen that the response varies quite a lot with the tilt of the anisotropy. The peak amplitude differs about 18 dB between the strongest and weakest cases. For a tilt of 30° and 60° the peak is not centred straight above the crack and the reason for this is that the group velocity (which defines the beam axis) is a little angled in these cases.

Figure 3 shows C-scans along a line from the 60° SV probe for a vertical square crack with side 4 mm and with a back surface at the centre distance 5 mm for four different tilts of the anisotropy. It is seen that the response varies quite a lot with the tilt of the anisotropy.
CONCLUSIONS

In this paper a complete model of ultrasonic testing is developed for a rectangular crack close to back surface in a thick anisotropic component. An arbitrary orientation of the anisotropy, the crack and the back surface are allowed. The model should be useful for physical understanding, parametric studies, etc. The numerical examples illustrates some of the possibilities.

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REFERENCES

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