THREE DIMENSIONAL BORN AND KIRCHHOFF INVERSIONS
FOR SHAPE RECONSTRUCTION OF DEFECTS

M. Yamada, K. Murakami, K. Nakahata, and M. Kitaharata
Department of Civil Engineering, Tohoku University
Aoba-yama06, Aoba-ku, Sendai, Miyagi 980-8579, Japan

ABSTRACT. Three dimensional Born and Kirchhoff inverse scattering methods are modified to convenient forms for a cylindrical specimen that includes three dimensional defects. The measurement area in the modified methods is restricted in the plane perpendicular to the axis of cylindrical specimen. The modified methods are applied to the cement paste specimen with an egg-shaped defect and the performance of the modified methods is confirmed.

INTRODUCTION

Geometrical characteristics of defects in structural components is a fundamental information to estimate the service life of the components. An approach for the characterization of the defect geometry is the imaging method based on the Born and Kirchhoff approximations. The linearized Born inversion method is studied in detail by Rose et al [1,2] and the Kirchhoff inversion method by Cohen and Bleistein [3,4]. A comprehensive review on the inversions is summarized by Langenberg [5]. Three dimensional elastodynamic inverse scattering methods are developed for the shape reconstruction of defects in solids [6,7]. In the three dimensional inversions, backscattered waveforms from all directions are required for the complete shape reconstruction. However, many civil structures like bridge piers are cylindrical objects and access points of transducers are limited to the surface of the cylinders. In this paper, three dimensional elastodynamic inverse methods in References [6,7] are modified to suitable form for the limited backscattering waveforms measured from the cylindrical surface. The performance of the modified method is investigated for the cement paste specimen with an egg-shaped artificial defect model.

THREE DIMENSIONAL LINEARIZED INVERSE SCATTERING METHODS

Three dimensional elastic body $D$ with the defect $D'$ is shown in Figure 1. The elastic modulus and the mass density are denoted by $C_{ijkl}$ and $\rho$ for the host material $D \setminus D'$, by $C_{ijkl} + \delta C_{ijkl}$ and $\rho + \delta \rho$ for the defect $D'$. The scattered wave field $u_m^s$ can be represented by
introducing the equivalent source $q_i(x)$ as

$$u^m_m(y) = \int_D G_m(x, y) q_i(x) dV$$  \hspace{1cm} (1)

where $G_m(x, y)$ is the fundamental solution.

In NDE application, the field point $y$ is usually far from the surface of the defect. Therefore we introduce the far-field approximation as shown in Figure 2:

$$|y - x| = |y - \hat{y} \cdot x|$$  \hspace{1cm} (2)

where $\hat{y} = y/|y|$ is the unit vector points to the observation point $y$ from the origin of the coordinate system. Introduction of the far-field approximation in the fundamental solution leads to the far-field approximation of the fundamental solution

$$G_m(x, y) \approx \frac{1}{\mu} \left[ D(k_T |y|) e^{ik_T \hat{y} \cdot x} + D(k_T |y|) (\delta_{m} - \hat{y} \delta_m) e^{-ik_T \hat{y} \cdot x} \right]$$  \hspace{1cm} (3)

where $D(z) = e^{z^2/(4\pi z)}$ and $\kappa = k_L/k_T = \sqrt{(1 - 2\nu)/(2(1 - \nu))}$.

The scattered wave field $u^m_m$ in Equation (1) can be expressed as follows

$$u^m_m(y) = A_m(k_L, \hat{y}) D(k_L | y |) + B_m(k_T, \hat{y}) D(k_T | y |)$$  \hspace{1cm} (4)

where $A_m$ and $B_m$ are scattering amplitudes at the far-field for longitudinal and transverse waves, respectively.

In this study, we consider the longitudinal-longitudinal pulse-echo configuration in measurement. The incident wave is assumed to be a plane longitudinal wave

$$u^0(x) = u^0 \hat{d}^0 \exp(i k^0 \hat{p}^0 \cdot x)$$  \hspace{1cm} (5)

where $u^0$ is the amplitude, $\hat{d}^0$ is the unit polarization vector, $k^0$ is the wave number of the incident wave, and $\hat{p}^0$ is the unit propagation vector. For the longitudinal wave, $k^0 = k_L$, $\hat{d}^0 = \hat{p}^0 = -\hat{y}$. The longitudinal scattering amplitude $A_m$ can be represented in the following two ways by using two types of equivalent source $q_i(x)$ in Equation (1). One is the volume type and the other is the surface type. For the volume type representation, $A_m$ is expressed as

$$A_m(k_L, \hat{y}) = \frac{1}{\mu} k^2 \gamma \delta_m \int_D \Gamma(x) (\delta \rho u^2 u_i(x) - \delta C_{ijkl} u_{ij}(x) \partial_i \partial_j x) e^{-ik_T \hat{y} \cdot x} dV$$  \hspace{1cm} (6)
where $\Gamma(x)$ is the characteristic function of the defect region $D^c$ defined by

$$\Gamma(x) = \begin{cases} 1 & \text{for } x \in D^c, \\ 0 & \text{for } x \in D \setminus D^c. \end{cases} \quad (7)$$

For the surface type representation, $A_m$ has the form

$$A_m(k_L, \hat{y}) = \frac{1}{\mu} \chi^{2} \gamma \psi_{m} \int_{D} \gamma(x) C_{ijkl}(n_i(x) u_k(x) - n_i(x) u_k(x) \partial / \partial x_j)e^{-ik_Lx} dV \quad (8)$$

where $\gamma(x)$ is the singular function which has values only on the defect surface $S$ and it is defined as

$$\int_{D} \gamma dV = \int_{S} dS \quad (9)$$

For a given $A_m$, both of the geometrical function ($\Gamma$ or $\gamma$) and the displacement field $u_m$ are unknown in Equations (6) and (8). Two types of approximations are now introduced for displacement fields in Equations (6) and (8).

**Born Inversion**

The Born approximation is to replace the displacement field $u$ in the defect $D^c$ by the incident wave $u^0$ in Equation (5). For a cavity, the elastic modulus $\delta C_{ijkl}$ and the mass density $\delta \rho$ are set to be $-C_{ijkl}$ and $-\rho$. Then the longitudinal scattering amplitude in Equation (6) is reduced to the following form

$$A_m(k_L, \hat{y}) = 2u^0 \chi^{2} \psi_{m} k_L^2 \int_{D} \Gamma(x)e^{-ik_Lx} dV \quad (10)$$

where $K = 2k_L \hat{y}$. The integral in the above equation is the Fourier transform, $\mathcal{F}(\Gamma(x)) = \hat{\Gamma}(K)$, of the characteristic function in the $K$ space. From the backscattered longitudinal waves in the frequency domain, we can obtain the scattering amplitude $A_m(k_L, \hat{y})$ and thus the function $\hat{\Gamma}(K)$ in the $K$ space. The characteristic function $\Gamma(x)$ is reconstructed from the inverse Fourier transform

$$\Gamma(x) = \frac{1}{(2\pi)^2} \int_{0}^{2\pi} \int_{0}^{\infty} \frac{\hat{\psi}_{m}}{2u^0 k_L^2} A_m(k_L, \theta, \phi) e^{iKx} dK \quad (11)$$

The characteristic function $\Gamma(x)$ reconstructs the inside of the defect in the three dimensional elastic body.

**Kirchhoff Inversion**

The Kirchhoff approximation is to replace the displacement field $u$ on the defect surface by the incident wave $u^0$ and the reflected waves. Then we can obtain the following expression for the longitudinal scattering amplitude from Equation (8)

$$A_m(k_L, \hat{y}) = 2u^0 \chi^{2} \psi_{m} k_L \int_{D} \gamma(x)e^{-ik_Lx} dV \quad (12)$$
The singular function $\gamma(x)$ is obtained as the inverse Fourier transform of the scattering amplitude

$$\gamma(x) = \frac{1}{(2\pi)^3} \int \int \int \frac{\hat{S}_m}{2i\mu k_L} A_m(k_L, \mathbf{y}) e^{iK \cdot x} dK$$

$$= \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^\infty \int_0^\infty \frac{\hat{S}_m}{2i\mu k_L} A_m(k_L, \theta, \phi) e^{2i\mu k_L y} x k^2 \sin \theta dk_L d\theta d\phi . \quad (13)$$

The singular function $\gamma(x)$ reconstructs the defect surface.

**Numerical Results from All Directions**

An ellipsoidal cavity with the aspect ratio $b/a = 0.5$ exists in the infinite three dimensional elastic body as shown in Figure 3. The incident longitudinal plane wave is sent to the ellipsoidal cavity and the backscattered longitudinal scattering amplitudes ($0.1 < a k_L < 10.0$) are calculated by the boundary element method. The Poisson’s ratio is set to be 0.345 (Aluminum) in the calculation. The scattering amplitudes from all directions are used for the three dimensional shape reconstruction formulas in Equations (11) and (13).

The results of shape reconstructions are shown in Figure 4. The left-hand side of Figure 4 is the result of Born inversion and the right-hand side is the result of Kirchhoff inversion. The Born inversion reconstructs the inside of the defect and the Kirchhoff inversion reconstructs the boundary. It is understood that the scattering amplitudes from all directions reconstruct the shape of defect well.

**SHAPE RECONSTRUCTION FROM RESTRICTED DIRECTIONS**

In the field, it is often difficult to measure scattered waves from all directions. Now, we consider the cylindrical structure that includes three dimensional defects in it as shown in Figure 5. In this type of structures, the ultrasonic measurement is restricted in the $x_1-x_2$ plane ($\theta = \pi/2$). In this case, the scattering amplitude $A_m(k_L, \theta, \phi)$ is set to be $A_m(k_L, \theta, \phi)\delta(\theta - \pi/2)$ in Equation (11). Then Equation (11) is reduced to the following form

$$\Gamma(x) = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^\infty \int_0^\infty \frac{\hat{S}_m}{2i\mu k_L} A_m(k_L, \theta, \phi) e^{2i\mu k_L x} x k^2 \sin \theta dk_L d\theta d\phi$$
In the same way, the expression for the singular function in Equation (13) is reduced to the following form:

\[
\gamma(x) = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^n \frac{4k_L}{iu_0} \hat{A}_m(k_L, \theta, \phi) \hat{\delta} \delta(\theta - \pi/2) e^{2ik_Lx} x 8k_L^2 \sin \theta dk_L d\theta d\phi
\]

Equations (14) and (15) are used to calculate the scattering amplitudes in the \(x_1-x_2\) plane from a spherical cavity with radius \(a\) in the range of wave numbers \(0.1 \leq k_L \leq 10.0\) as shown in Figure 6. The calculated amplitudes are applied to the modified three dimensional shape reconstruction formulas in Equations (14) and (15). The results of shape reconstructions are shown in Figure 7 as the cross sectional view in the \(x_1-x_2\) plane. The shape of the defect is reconstructed fairly well from the scattering amplitudes measured in the \(x_1-x_2\) plane.
EXPERIMENTAL MEASUREMENT

A cement paste cylinder with the diameter 150mm is prepared as shown in Figure 8. In the cylinder, an egg-shaped foam polystyrene is inserted as an artificial defect. The specimen for the reference wave measurement is also shown in Figure 8. The experimental setup is shown in Figure 9.

The experiments were carried out in the water tank equipped with the PC-controlled manipulator and turntable. The scattered waveforms from the defect model are measured by the longitudinal – longitudinal pulse echo method with the immersion type transducer whose center frequency is 1.0MHz. The measurement $x_1$-$x_2$ plane was adjusted to get the largest signal from the defect. The cement paste specimen is fixed on the turntable and turns 10 degrees a step in the $x_1$-$x_2$ plane and 36 waveforms are recorded on the digital oscilloscope as the time-averaged data. The data in time domain are transformed to the frequency domain.

Data Processing

The longitudinal scattering amplitude $A_m$ in the solid is required for the shape reconstruction. To extract the scattering amplitude $A_m$, the data processing with the reference waveform is adopted here [8]. We prepared the specimen for the reference measurement as shown in the right-hand side of Figure 8. The reference specimen has a planar free surface.
in it and it is made of the same cement paste as the specimen with the defect model. The received waveform $O^{sc}(f)$ in the experimental system is expressed in the frequency domain

$$O^{sc}(f) = I(f)T(f)W(f)H_{ws}(f)E^{sc}(f)H_{sw}(f)W(f)R(f)$$

(16)

where $I(f)$ is the input-signal, $T(f), W(f), H_{ws}(f), H_{sw}(f), R(f)$ are the effects of the transmitting transducer, water path, transmission from water to solid, transmission from solid to water, receiving transducer, respectively. The term $E^{sc}(f)$ represents the scattering effect in the solid. The reference signal $O^{ref}(f)$ is written as

$$O^{ref}(f) = I(f)T(f)W(f)H_{ws}(f)E^{ref}H_{sw}(f)W(f)R(f)$$

(17)

where $E^{ref}$ is a constant and equal to the reflection coefficient at a planar free surface. The measured signal in Equation (16) is the same as the reference signal in Equation (17) except for terms $E^{sc}$ and $E^{ref}$. Therefore the scattered waveform in the cement paste is obtained from

$$E^{sc}(f) = E^{ref} \frac{O^{sc}(f)}{O^{ref}(f)}.$$  

(18)

Examples of measured waveform $O^{sc}(t; \phi=90^\circ)$ and the reference waveform $O^{ref}(t)$ are shown in Figure 10 and Figure 11. The processed waveform $E^{sc}(f)$ can be used as the longitudinal scattering amplitude $\hat{y}_{x_{m}}A_{n}(k_{1}, \pi/2, \phi)$ in Equations (14) and (15).

**Shape Reconstruction from Experimental Data**

$\Gamma(x)$ and $\gamma(x)$ reconstructed from scattering amplitudes obtained by the experiment are shown in Figure 12. In Figure 12, the left-hand side represents the result of the Born inversion and the right-hand side the Kirchhoff inversion in the $x_1-x_2$ plane. After the measurement, the specimen was cut into halves in the $x_1-x_2$ plane. The cross sectional photograph is shown in Figure 13. The egg-shaped boundaries by $\Gamma(x)$ and $\gamma(x)$ have good agreement with real boundary of the defect model.
CONCLUSIONS

Three dimensional linearized inverse scattering methods are modified to convenient form to use waveforms observed from the cylindrical surface. Modified Born and Kirchhoff inversions reconstruct the shape of defect in the limited plane from measurement data obtained for the cylindrical specimen.

REFERENCES