On the Numerical Simulation of Rotating Rarefied Flow in the Cylinder with Smooth Surface

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Abstract. Numerical simulation of impulsively started rotational flow in the cylinder with smooth surface was carried out using the DSMC method. Results of present simulation affirmed that an isothermal solid rotation of the fluid was the thermally and mechanically equilibrium state of the rotating fluid. Integrating mass, angular momentum, and energy conservation equations, we obtained area-angular velocity relation of the rotating fluid (vortex filament) under compression or under expansion. We found that speed ratio (Mach number) of the peripheral velocity on the surface of the cylinder changed very slightly as the fluid was compressed or expanded so long as the peripheral Mach number was low or moderate subsonic. Thus, theoretical results suggested that most part of the work done by the compression of the vortex filament reduced to the internal energy of the fluid and the ratio of the energy of rotating fluid to the internal energy, \( E_{\text{rot}}/\epsilon \approx s^2/3 \approx \text{constant.} \)

These theoretical predictions were affirmed by the numerical simulation of DSMC method, where the steadily rotating fluid (vortex filament) was compressed by contracting the cylinder. Adiabatic compression of the vortex filament yielded another isothermal solid rotation of the fluid and the ratio of the energy to the internal energy, \( E_{\text{rot}}/\epsilon \), was almost unchanged.

INTRODUCTION

This paper was concerned generation of rotating flow and contraction of the rotating flow. Here, the rotating flow in the cylinder with smooth surface was considered as a model of two-dimensional vortex or core of the vortex. Contraction of the cylinder was, accordingly, regarded as the contraction of the vortex filament. Vortex filaments with various vorticity distributions were often introduced in the Eulerian flow field and worked as the model of actual flow (See Saffman [1]). On the other hand, Karman vortex behind the cylinder was well studied experimentally and also studied numerically [2], [3]. In the numerical simulations flow field and vortex were solved simultaneously and flow field induced by the vortices were not examined. Thus, some gaps between actual vortices to the vortex model or model of vorticity distribution had to be bridged. In the numerical simulation of shock wave-vortex interaction [4], [5] Taylor’s vortex and Rankine’s vortex were employed. In these simulations vortex suffered compression by the shock wave and deformation of the vortex and acceleration or deceleration of the angular velocity of the vortex was treated. In the studies of two-dimensional turbulence [6] or in the studies of three-dimensional Taylor-Green vortex [7] concrete structures of the vortex filament or vortex ring were treated. Spectral method used in later cases did not require to know the structure of the vortex but to interpret the results understanding of the fundamental structure of the vortex filament or vortex ring must be necessary. First of all we were required to know equilibrium state of vortex; for the compressible flow as the case of shock-vortex interaction we needed to know mechanical equilibrium state and also thermal equilibrium state. Since the Navier-Stokes solver might usually include artificial viscosity, we were not sure whether momentum and energy of the flow were conserved precisely. To avoid such a uncertainty present simulation was carried out using the DSMC method where momentum and energy were conserved in the molecular level. Consequently, as the demerit, we could treat only small size vortexes. Except the

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nonlinear phenomenon such as the cascade of vertexes results of simulations might be applied to the cases of small Knudsen number. Existence of equilibrium solution was demonstrated and thermodynamic relations for the contracted vortex was shown in the next section.

AN EQUILIBRIUM ROTATING FLOW IN THE CYLINDER

In order to know the steadily rotating flow field we examined the equilibrium state of rotation. The steady state Boltzmann equation in the cylindrical coordinates \((z, r, \theta)\) (See Fig. 1) for monatomic gas is given by

\[
\frac{\partial f}{\partial t} + c_r \frac{\partial f}{\partial z} + c_r \frac{\partial f}{\partial r} + c_\theta \frac{\partial f}{\partial \theta} + \frac{c_r^2}{r} \frac{\partial f}{\partial c_r} - c_r c_\theta \frac{\partial f}{\partial c_\theta} = \int \int \int d\varepsilon \int \int (f' f'_1 - f f_1) g\sigma d\Omega \tag{1}
\]

where \(f = f(\varepsilon)\) denotes the velocity distribution function, \(\varepsilon = (c_z, c_r, c_\theta)\) the molecular velocity, \(g = |\varepsilon|\) the relative velocity, \(\sigma = \sigma(g, \chi)\) the collision cross section, \(\chi\) the deflection angle, \(d\Omega = \sin \chi d\chi d\epsilon\), \(\epsilon\) the angle of inclination of the plane of collision.

Substituting an equilibrium distribution function,

\[
f = f_0 = n(2\pi RT)^{-3/2} \exp \left[ \frac{c_z^2 + c_r^2 + (c_\theta - u_\theta)^2}{2RT} \right] \tag{2}
\]

into Eq. (1) and rearranging it, we obtain,

\[
\frac{\partial f_0}{\partial t} + c_r \left( \frac{1}{p} \frac{\partial p}{\partial r} - \frac{\rho u_\theta}{p} \frac{\partial u_\theta}{\partial r} \right) f_0 + c_r \left( \frac{C^2}{2RT} - \frac{5}{2} \right) \frac{1}{T} \frac{\partial T}{\partial r} f_0 + \frac{2c_r c_\theta}{2RT} \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) f_0 = 0 \tag{3}
\]

where we use the state equation, \(p = \rho RT\). Here, notations are conventional ones. Thus, for steady state rotation \(\partial/\partial t = 0\), the distribution function Eq. (2) with

\[
u_\theta = r\omega \quad (a), \quad T = \text{const.} \quad (b), \quad \frac{dp}{dr} = \rho r\omega^2 \quad (c) \tag{4}
\]

satisfies the Boltzmann equation exactly [8], [9]. Since on the smooth and adiabatic wall radial velocity, \(u_r = 0\), \(\partial T/\partial r = 0\), and \(r\theta/\partial r(u_\theta/r) = 0\) equations (4a) to (4c) satisfy the boundary conditions on the wall of the cylinder. Isothermal solid rotation is the equilibrium state of the rotating fluid in the cylinder with the smooth surface. So, the Rankine’s vortex (i.e., vortex filament with circular cross section) is an isothermal rotation of the fluids. Equation (4c)yields distributions of density and pressure in the solid rotation,

\[
\rho(r) = \rho(0) e^{r^2 \omega^2 / 2RT} \tag{5}
\]

\[
p(r) = p(0) e^{r^2 \omega^2 / 2RT} \tag{6}
\]
We assume that when we quasi-stationarily contract the surface of the cylinder and change the cross section of the cylinder the isothermal solid rotation continues. Temperature and angular velocity of the fluid may change gradually. When the cylinder (vortex filament) is contracted and the radius of the cylinder is decreased from $a_{st}$ to $a$ where the subscript "st" implies the steady state rotation before the contraction, the angular velocity and the temperature of rotating fluid (vortex filament) may be described in terms of the contraction ratio, $a/a_{st}$. Conservation equations of mass of the fluid in the cylinder (in the filament) can be expressed by,

$$M = \int_{0}^{a} \rho(2\pi rdr) = \pi a^2 \rho(0, s) \frac{1}{2} \left( \epsilon^2 - 1 \right) = \text{const}$$  \hspace{1cm} (7)

where $a$ denotes the radius of the cylinder (vortex filament) and $s$ denotes the speed ratio defined by $s^2 = a^2 \omega^2 / 2RT$ where $\omega = \omega(a)$. Since the density $\rho(r)$ depends upon the radius of the cylinder $a$ or $a/\sqrt{2RT}$ we denote as $\rho(r, s)$. In deriving Eq. (7), Eqs. (5) and (6) are used. Conservation equations of angular momentum of the fluid in the cylinder (in the filament). can be expressed by,

$$L = \int_{0}^{a} r\rho(2\pi rdr)u_\theta = 2\pi \omega \rho(0, s) a^4 \frac{s^2 \epsilon^2 - \epsilon^2 + 1}{2s^4} = \text{const}. \hspace{1cm} (8)$$

Equation (8) yields the ratio of angular velocity $\omega/\omega_{st}$,

$$\frac{\omega}{\omega_{st}} \left( \frac{a^2}{a_{st}^2} \right) = \left( \frac{s^2}{s_{st}^2} \right) \left( \frac{\epsilon^2 - 1}{\epsilon_{st}^2 - 1} \right) \left( \frac{s_{st}^2 \epsilon_{st}^2 - \epsilon_{st}^2 + 1}{s^2 \epsilon^2 - \epsilon^2 + 1} \right) \hspace{1cm} (9)$$

Conservation equation of energy of the fluid is given by,

$$M c_v dT = -pdV - \frac{1}{2}L d\omega. \hspace{1cm} (10)$$

The second term in the right-hand side of Eq. (10) denotes the work done in the fluid by the surface of cylinder. Substituting the relations,

$$p = \rho(0, s)\epsilon^2 RT, \hspace{1cm} dV = 2\pi a da \cdot 1 = \pi da^2, \hspace{1cm} \frac{1}{2}L d\omega = dE_{rot} = E_{rot, st} \frac{d\omega}{\omega_{st}},$$

into Eq. (10) and dividing with $M c_v T_{st}$ where $c_v$ denotes the specific heat at constant volume, we obtain

$$\frac{d(T/T_{st})}{da^2} + \left( \frac{E_{rot, st}}{M c_v T_{st}} \right) \frac{d(\omega/\omega_{st})}{da^2} = -\frac{\pi \rho(0, s) RT}{M c_v T_{st}} \epsilon^2$$  \hspace{1cm} (11)

The coefficient of the second term in the left-hand side of Eq. (11) implies the ratio of rotational energy to the internal energy and the ratio reduced to,

$$\frac{E_{rot, st}}{M c_v T_{st}} = (1/2)L_{st} \omega_{st} / M c_v T_{st} = (1/2)((1/2)M a_{st}^2 \omega_{st}) / M(3/2)RT_{st} = \frac{1}{3}s_{st}^2 \hspace{1cm} (12)$$

Since $T/T_{st} = (a\omega/a_{st}\omega_{st})^2 (s_{st}/s)^2$, the derivative of the temperature $d(T/T_{st})$ can be expressed in terms of $d(\omega/\omega_{st})$ and $d(s^2/s_{st}^2)$, while $d(\omega/\omega_{st})$ can be replaced by $d(s^2/s_{st}^2)$ with the aid of Eq. (9). Taking into account these relations, Eq. (11) can be rewritten as

$$\frac{dq}{d\sigma} = -\frac{3}{2} \frac{s q e^q - \frac{1}{3} G(q)}{\sigma F(q)}$$  \hspace{1cm} (13)

$$\begin{cases}
F(q) = e^q + \frac{2}{3}q + 1 + \frac{2}{3}e^q - 1 + \frac{4}{3}e^q - \frac{1}{q} - \frac{2}{q e^q - e^q + 1} \hspace{1cm} (14)
\end{cases}$$

where $q = a^2 \omega^2 / 2RT$ and $\sigma = a^2/a_{st}^2$. Substituting the solution of Eq. (13) into Eq. (9) we obtain the angular velocity $\omega$, while the temperature is determined by $T/T_{st} = \sigma(q_{st}/q)(\omega/\omega_{st})^2$. Results of Eq. (13) will be discussed later.
PROCEDURE OF SIMULATION

Monte Carlo simulation of rotating flow in the cylinder and compression of the rotating flow is executed in this paper. As shown in Fig. 2 a rest gas in the cylinder is impulsively rotated in a uniform angular velocity, $\omega_0$. Centrifugal force acted on the fluid generates radial velocity and unsteady pressure waves propagate outward and inward repeatedly. After several hundred or several thousand mean free times we can expect to obtain the steadily rotating flow described by Eq. (4); resulting flow may be in mechanically equilibrium and also in thermal equilibrium. Once such a steady state is obtained, the wall of the cylinder is made to move inward and it begins to compress the rotating fluid. Since the molecules reimpinged on the surface of the cylinder specularly on the surface, angular momentum around the axis of symmetry must be unchanged. Through the molecular collisions gas in the cylinder approaches to an equilibrium state as expected by Eq. (4). Data of simulation is listed in Table 1.

A Generation of vortex filament: At $t = 0$ the peripheral velocity of a molecule located at $r = r_j$ is increased from $v_0$ to $v_0 + r_j\omega_0$ where $v_0$ is the velocity that is generated stochastically around $v_0 = 0$. Then molecular gas in the cylinder impulsively starts to rotate. Since the molecules reflect specularly on the surface, angular momentum around the axis of symmetry must be unchanged. Through the molecular collisions gas in the cylinder approaches to an equilibrium state as expected by Eq. (4). Data of simulation is listed in Table 1.

<table>
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<th>TABLE 1. Properties used in the simulation of vortex generation</th>
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(m.f.t.: mean free time , Interval: Sampling interval)
B Compression of vortex filament: The generated steady state rotating fluid (vortex filament) is gradually compressed by contracting the radius of the filament (the radius of the cylinder). The radius of the cylinder is decreased in the rate of $1\lambda_0/240$ in each time increment $\Delta t_m = 1/20$ mean free time. This means that the inward velocity of the wall of cylinder is about $0.1/\sqrt{2RT}$; this velocity must be too fast to realized a quasi-stationary compression. Molecular data of the last time step of each ensemble element in the simulation of vortex-generation were employed as the initial data of compressing process. The radius $a$ was contracted to 80% of the initial radius $a_0$ and the cross section was contracted to 64% of the initial cross section. After the contraction of the radius was finished simulation was continued until a steady state rotation might be formed in the cylinder with the contracted radius. Such a simulation was carried out for many ensembles. Data of simulation are listed in Table 2.

**TABLE 2.** properties used in the simulation of compression of vortex filament

<table>
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<tr>
<th>$a_0$ ($\times \lambda_0$)</th>
<th>rate of contraction ($a_{st}/a_0$)</th>
<th>$\Delta t_m$ (in m.f.t.)</th>
<th>Interval contracting duration (step)</th>
<th>Ensemble in contrac. after contrac.</th>
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<tr>
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<td>$1\lambda_0/240$</td>
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(m.f.t.: mean free time, Interval: sampling interval)

**FIGURE 3.** Temporal variation of temperature; DSMC(left and center), N-S Equation(right) (vortex generation)

**RESULTS AND DISCUSSION**

**A Generation of vortex filament**

As shown in Table 1 we carried out simulation for three cases, $Kn = 0.05, 0.025$, and $0.01$. For the case of $Kn = 0.05$ we obtained a steadily rotating flow within 100 mean free time after the impulsive rotation started. It took about 300 mean free time to reach the steady state rotation for the case of $Kn = 0.025$ (See Fig. 3, left). On the other hand, for the case of $Kn = 0.01$, oscillation of pressure and temperature in the fluid continue long time (See Fig. 3, center) while the peripheral velocity of the fluid in the cylinder showed a faster convergence to a steadily solid rotation (See Fig. 4, left). Long time behavior of the oscillation of temperature was examined using the Navier-Stokes equation. Figure 3(right) showed that oscillation was decaying gradually and it would take several thousand mean free time to reach a steady state. The results at 9500 time steps (450 mean free time after the impulsive start of rotation) (See Fig. 4, center and right) showed temporal distribution of pressure and temperature but as shown in Fig. 3(right) and Fig. 4(left) rotating fluid in the cylinder would approach an isothermal solid rotation. Angular velocity decreased to $\omega = 0.967\omega_0$ and then speed ratio decreases to $s = 0.967s_0 = 0.484$. Increase of temperature was about 0.6%. Decrease of the rotational energy was $\Delta E_{rot}/E_{rot0} = 0.063$ and the expected increase of the temperature was $c_vT/c_cT_0 = 0.063 \times s_0^2/3 = 0.52\%$. Present results showed that the peripheral velocity $u_0$ in the vicinity of outermost cells deviated from the distribution of solid rotation for the cases of $Kn = 0.05$ and $0.025$. Since the number of molecules positioned in $r > r_{cellcenter}$ might be larger than the number positioned in...
molecular collisions made the angular momentum around the center of cell decreased. So long as the ratio of the cell width to the radius of the cylinder, \( \Delta r_0/a_0 \) was negligibly small, such a decrease of angular momentum might become nonnegligible as the time steps of simulation proceeded.

**B Compression of vortex filament**

Using the date of the simulation of vortex generation, the boundary of the outermost cell was moved inward with the speed \( u_r = 0.1\lambda_0/1.2t_c \) until \( a/a_0 = 0.8 \). Resulting ratio of cross section was \( \sigma = a^2/a_0^2 = 0.64 \). As aforementioned the rate of contraction was too high, movement of the wall exerted pressure wave as shown in Fig. 5 and the waves continued after contraction was finished. As the time proceeded radial fluid motions gradually decayed out and compressed fluid formed a steady rotation. In the case for the Kn = 0.01, however, damping of the waves were very slow and the oscillation of the pressure continued over 1,250 mean free times. Figure 6 showed temporal and spatial variations of the temperature. Temperature of the rotating fluid was almost uniform and we found that temperature increased to \( T \approx 1.356 \sim 1.365T_0 \). Quasi-stationary compression increased the temperature up to \( T = 1.597T_0 \). This implied that compression was not carried out effectively like the compression by the shock waves. Figure 7 showed distributions of peripheral velocity, pressure and temperature of the contracted vortex (compressed rotating flow) after 20,000 time steps (1,000 mean free time) after the contraction was finished. They showed a isothermal solid rotation. As aforementioned the maximum speed ratio pertinent to the peripheral velocity changed slightly as the cross section of the vortex decreased (See Fig. 8). Solution of Eq. (13) (Fig. 8) yielded \( s(a = 0.8a_0) = 0.486 \) while, before the compression, \( s(a_0) = 0.484 \). In Fig. 9 were shown increase of internal energy and increase of energy of rotation. Since \( E/E_0 \approx 1.36 \) and \( E_{rot}/E_{rot0} \approx 0.11 \), the ratio \( E/E_{rot} = 0.08 = s^2/3 \). Thus, we obtained \( s = 0.493 \) and agreed well with the predicted value 0.483.

Through the present simulation we revealed that the isothermal solid rotation was the equilibrium state of the
rotating flow and the energy included in the rotating motion of the flow was well estimated by $s^2/3$. Actual vortexes or model vortexes are composed of the vortex core and the surrounding potential vortex or vortex with negative vorticity. However, so long as the vortex sustained long time and vortex core held same geometry, vortex core must approach the isothermal solid rotation and $s^2/3 = \left(\frac{a^2 \omega^2}{\sqrt{2RT}}\right)/3$ must be a convenient measure of the energy of vortex.

CONCLUDING REMARKS

Numerical simulation of impulsively started rotational flow in the cylinder with smooth surface was carried out using the DSMC method. Results of present simulation affirmed that an isothermal solid rotation of the fluid was the thermally and mechanically equilibrium state of the rotating fluid. Integrating mass, angular momentum, and energy conservation equations, we obtained area-angular velocity relation of the rotating fluid (vortex filament) under compression or under expansion. We found that speed ratio (Mach number) of the peripheral velocity on the surface of the cylinder changed very slightly as the fluid was compressed or expanded so long as the peripheral Mach number was low or moderate subsonic. Thus, theoretical results suggested that
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These theoretical predictions were confirmed by the numerical simulation of DSMC method, where the steadily rotating fluid (vortex filament) was compressed by contracting the cylinder. Adiabatic compression of the vortex filament yielded another isothermal solid rotation of the fluid and the ratio \( \frac{E_{\text{rot}}}{c_v T} \) was almost unchanged.

REFERENCES