Phoresis in a Shearing Gas

Lars H. Söderholm and Karl I. Borg

Department of Mechanics
Royal Institute of Technology
SE-100 44 Stockholm, Sweden

Abstract. An axially symmetric body small compared with the mean free path is free to move in a shearing gas. The body is treated as a test particle. The force and torque acting on the body are calculated. This force and torque will set the body in motion, which asymptotically will take place in one of the eigendirections of the rate of deformation tensor. The axis of the body then points in the same direction. For a velocity field \( v_x(y) \) the final motion is parallel to one of the lines \( x = y \) and \( x = -y \), and the speed of the motion is given by

\[
V = \frac{9 \mu \beta_N}{8 p} \left( \frac{2 \pi k_B T}{m} \right)^{1/2} \frac{\alpha_r b_1}{4 + \frac{1}{2} \pi \alpha_r + [8 - (6 - \pi \alpha_r)] b_3} \nu_{x,y}.
\]

Here \( \mu \) is the viscosity of the gas, \( p \) is the pressure, \( \beta_N \) is a number close to unity, \( T \) is the temperature, \( m \) is the mass of a gas molecule, and \( \alpha_r \) is parameter in the boundary conditions close to unity. The non-dimensional numbers \( b_1 \) and \( b_3 \) depends on the shape of the body. This speed is of the order of the mean free path of the gas multiplied by the shearing. There will be no motion for a body, which is reflection symmetric in a plane orthogonal to the axis of symmetry. This means that there is a phenomenon of phoresis in a shearing gas, which is analogous to thermophoresis in a gas with a temperature gradient.

INTRODUCTION

For small Knudsen numbers, there are two basic deformations of the local Maxwellian. One from a temperature gradient and one from shearing. In an earlier paper of one of the authors, [5], the influence of shearing on the heat transfer to a body was studied. It was found there, that due to the tensorial nature of the shearing, shearing did not influence the heat transfer to a spherical body, so bodies of arbitrary symmetry were considered. It was found that the equilibrium temperature of an axially symmetric small body was influenced by the shearing and thus was different from that of the surrounding gas.

In an other work by the present authors, [6], the idea of axially symmetric bodies was applied to the phenomenon of thermophoresis and the full rigid body motion of such a small body was studied.

In this work, the influence of shearing on axially symmetric bodies is studied. Force and torque on the body exerted by the surrounding gas are calculated. It is found that there is a motion, a phoresis, in a shearing gas, which is analogous to that of thermophoresis. It vanishes for bodies, which are mirror symmetric with respect to a plane perpendicular to the axis of symmetry of the body, in particular for spheres. We name this motion shearing phoresis.

In this work we shall approach the problem in the way outlined by Waldmann in [3].

THE DISTRIBUTION FUNCTION OF THE SHEARING GAS

We consider a convex high conductivity body, small compared to the mean free path of the gas. The gas is subjected to a velocity gradient. As a consequence the small body will start to move. This means that the motion of a particular point in the body will be given by \( \mathbf{u'} = \mathbf{u} + \omega \times \mathbf{x} \), where \( \mathbf{u} \) is the velocity of the body's center of mass, \( \omega \) is the angular velocity of the body and \( \mathbf{x} \) is the vector from the body's center of mass to the particular point we are considering. As the interaction between the gas and the body is mediated through the body surface, it is convenient to take the motion of a body surface element into account by transforming the
distribution function to a frame of reference where the surface element is momentarily at rest. In this frame, the gas is flowing with the velocity \(-u\). To first order in the mean free path, the distribution function for the gas molecules can then be written \[4\]

\[ f = f_0(-u) (1 + \phi_{\text{shear}}), \]  

where \(f_0(-u)\) is a Maxwellian moving with the macroscopic velocity \(-u\), and where \(\phi_{\text{shear}}\) is the first-order Chapman-Enskog correction to the Maxwellian for a shearing gas. If the velocity of the body is small compared to the speed of sound, \(f_0(-u)\) can be linearized according to \(f_0(-u) \approx f_0(0)(1 - \frac{m}{k_B T} c \cdot u')\). This assumption is consistent with our results for the final speed of the body. \(f_0(0)\) is the Maxwellian describing the gas at rest. In what follows, we will for simplicity denote \(f_0 \equiv f_0(0)\). Under these circumstances we can also omit the cross-effect between the shearing and homogeneous flow. If we introduce the non-dimensional molecular velocity \(C_i \equiv \sqrt{m/2k_B T_c}\), (here \(m\) is the mass of a gas molecule, \(k_B\) is Boltzmann’s constant and \(T\) the temperature) the Maxwellian is given by

\[ f_0 = n \left( \frac{2\pi k_B T}{m} \right)^{-3/2} \frac{1}{e^{c^2}}. \]

\(n\) is the number of molecules per unit volume. Thus, the total distribution function is linearized around the Maxwellian \(f_0\) and is expressed by

\[ f = f_0 (1 + \phi_{\text{shear}} + \phi_{\text{flow}}). \] 

Here \(\phi_{\text{flow}} = -\frac{m}{k_B T} c \cdot u'\) is the contribution from the homogeneous flow, and \(\phi_{\text{shear}}\) has the explicit form \[4\]

\[ \phi_{\text{shear}} = -\frac{2\mu}{p} B(C^2) C_{i<j} v_{<i,j>}. \] 

In this expression, \(v_i\) is the velocity field of the gas. Further, \(<..>\) denotes the symmetric and traceless part. The scalar function \(B(C^2)\) is usually expanded in Sonine polynomials. It is normalized so that \(B = 1\) if only the first term in the Sonine-expansion is retained.

**THE FORCE EXERTED BY THE GAS ON A BODY SURFACE ELEMENT**

We shall now calculate a general expression for the net force exerted by the surrounding gas described by \(f\), the distribution function of the gas in the absence of the body, on a resting surface element \(dS_i = n_i dS\). Here \(n_i\) is the unit normal of the surface element. The object of this section is to express this force completely in terms of \(f\).

The net force exerted on the surface element is the difference of the momentum carried in to the surface element by the incident stream of gas molecules and the momentum carried out by the reflected stream, and can be written

\[ dF_k = \left[ P_k^{(i)} - P_k^{(r)} \right] dS, \]  

where the momentum flux incident on the surface element \(P_k^{(i)}\) is given by

\[ P_k^{(i)} = -\int_{c_i n_j < 0} m c_k c_j n_j f^{(i)} d^3c. \]  

If the surrounding gas is described by a resting Maxwellian distribution, (5) takes the value \(-P_M n_i\), where

\[ P_M = \frac{1}{2} nk_B T. \]  

The momentum flux carried out by reflected molecules \(P_k^{(r)}\) is given by

\[ P_k^{(r)} = \int_{c_i n_j > 0} m c_k c_j n_j f^{(r)} d^3c. \]
Here, \( f^{(i)} \) and \( f^{(r)} \) are the distribution functions of the stream of molecules incident and reflected on the surface element. Since the body is small compared to the mean free path of the gas, and as the body is convex, the incident stream of molecules can be approximated by the distribution function describing the gas in the absence of the body, that is, \( f^{(i)} = f \). The reflected stream of gas molecules is given by Maxwell’s boundary condition, cf Kogan [1]. This means that the reflected stream has two separate parts: One part is specularly reflected (that is, reflected as a particle hitting a solid wall). The remaining part of the reflected stream has reached thermal equilibrium with the surface, and is reflected as Maxwellian. Thus we have

\[
f^{(r)} = (1 - \alpha_r) f^{(i)} (c - 2(c \cdot n)n) + \alpha_r n^{(w)} (\frac{2\pi k_B T^{(w)}}{m})^{-\frac{3}{2}} \exp \left(-\frac{mc^2}{2k_B T^{(w)}}\right).
\]

The number \( \alpha_r \) is called the accommodation coefficient of tangential momentum, and measures the fraction of the reflected stream that is diffusely reflected. \( T^{(w)} \) is the temperature of the surface of the body. We will set \( T^{(w)} = T \). One can show that this is a good approximation for bodies of high thermal conductivity. The unknown number density \( n^{(w)} \) can be determined from the condition that the net particle flux through the surface vanishes. We now define the incident particle flux according to

\[
N^{(i)} = -\int_{c_j n_j < 0} c_j n_j f^{(i)} d^3 c.
\]

For a Maxwellian distribution at rest \( f^{(i)} = f^{(0)} \), (9) takes the value

\[
N_M = n \sqrt{\frac{k_B T}{2\pi m}}.
\]

We are now in a position to calculate the net momentum flux transferred to the surface element according to (4). It is easy to see that in case of pure specular reflexion, the momentum transferred to the surface element is given by \( 2nkHjPj \). The force on the surface element (4) becomes

\[
dF_k = \left[ 2(1 - \alpha_r) n_k n_j P_{j}^{(i)} + \alpha_r \left( P_{k}^{(i)} - \frac{N^{(i)}}{N_M} P_M n_k \right) \right] dS.
\]

The first term is the momentum transferred to the body from the specularly reflected stream. The second term, \( \alpha_r P_{k}^{(i)} \), is the part of the incident momentum to be reflected diffusely. The last term is the momentum carried out from the surface by the diffusely reflected stream. Note that \( N_M \) and \( P_M \) only depends on the gas parameters.

**FORCE AND TORQUE ON A SURFACE ELEMENT**

The fluxes \( N^{(i)} \) and \( P^{(i)} \) are now calculated for the distribution function (2), with \( \phi_{\text{flow}} = -\frac{m}{k_B T c \cdot u'} \), and with \( \phi_{\text{shear}} \) given by (3). The resulting expressions are then substituted into (11), and the result is three different contributions to the force on the surface element of the body (here, \( p = nk_B T \)):

\[
dF_i = -pm_i dS
+ \frac{1}{2} \mu \alpha_r \beta_N n_{i<j} n_{k>v} u_{i<j,k>v} dS.
\]

\[- \left( \frac{2\pi k_B T}{m} \right)^{-\frac{1}{2}} p \left\{ \alpha_r u_i' + \left[ 4 - \left( 3 - \frac{\pi}{2} \right) \alpha_r \right] u_j' n_j n_i \right\} dS.
\]

The first force stems from the resting Maxwellian, the second from the shearing and the third from the homogeneous flow. In the expression for the force due to the shearing, the number \( \beta_N \) is given by

\[
\beta_N \equiv \int_0^\infty C^5 \tilde{B}(C^2) e^{-C^2} dC = \frac{1}{2} \int_0^\infty x^2 \tilde{B}(x) e^{-x} dx.
\]

If \( \tilde{B}(x) \) is expanded in Sonine polynomials and only the first term is retained, \( \beta_N = 1 \).
FORCE AND TORQUE ON THE BODY

The object of this section is to integrate the force over the surface of an axially symmetric body, for which we denote the axis of symmetry \( N \). In addition, the gas will in general also produce a torque acting on the body according to (here \( \epsilon_{ijk} \) is the totally antisymmetric permutation pseudo tensor, with \( \epsilon_{123} = 1 \))

\[
dM_i = \epsilon_{ijk} x_j dF_k.
\]

(14)

Here, \( x \) is the point on the surface element with the normal \( n \), with respect to the center of mass of the body.

The force due to the shearing becomes, after integration of expression (12) over the surface of the body

\[
F_{(\text{shear}) i} = -\frac{1}{12} \mu \alpha \beta S b_1 (\delta_{ij} - 15N_{<i}N_{j>}) v_{<j,k>} N_k,
\]

(15)

where

\[
b_1 = \frac{1}{S} \int_S (n \cdot N)^3 dS.
\]

(16)

Note that \( b_1 = 0 \) if the body is mirror symmetric.

The torque due to the shearing (14) is integrated over the body surface, and the result is

\[
M_{(\text{shear}) i} = \frac{1}{2} \mu \alpha \beta S^3/2 b_2 \epsilon_{ijk} N_j v_{<k,i>} N_i.
\]

(16)

The coefficient \( b_2 \) is a scalar integral given in appendix. Note that in contrast to \( b_1 \), \( b_2 \) does not in general vanish if the surface is mirror symmetric in a plane orthogonal to the axis of symmetry.

We will now calculate the contributions from the flow. Here we transform to the frame where the gas is at rest, and in which the body translates and rotates. The force and the torque are invariants under this transformation. In terms of the velocity of the body’s center of mass \( u \) and the angular velocity of the body, \( \omega \), we get (c.f. [6])

\[
F_{(\text{flow}) i} = \left( \frac{2 \pi k_B T}{m} \right)^{-1/2} \left[ -pS (a_1 \delta_{ij} + a_2 N_{<i}N_{j>} - 3p S^{3/2} a_3 \epsilon_{ijk} N_j) u_j + 3p S^{3/2} a_3 \epsilon_{ijk} N_j \omega_k. \right]
\]

(17)

Here, the coefficients \( a_1, a_2, a_3 \) are given in the appendix.

The corresponding torque is given by (c.f. [6])

\[
M_{(\text{flow}) i} = \left( \frac{2 \pi k_B T}{m} \right)^{-1/2} \left\{ -3p S^{3/2} a_4 \epsilon_{ijk} N_j v_k - \frac{p S^2}{2} (a_5 \delta_{ij} + a_6 N_{<i}N_{j>}) \omega_j \right\}.
\]

(18)

The coefficients \( a_4, a_5, a_6 \) are listed in the appendix.

EQUATIONS OF MOTION

To formulate the equations of motion of the system, we introduce the body mass \( m_B \) and the tensor of inertia, \( I_{ij} \). This is given by

\[
I_{ij} = m_B S (I_1 \delta_{ij} + I_2 N_{<i}N_{j>}).
\]

(19)

Here the coefficients \( I_1, I_2 \) are dimensionless moments of inertia. They are given in Appendix. Further, we introduce a body-fixed orthonormal frame of reference with the origin in the body’s center of mass. The basis vectors are denoted by \( e^{(i)}_\alpha \) where the index \( \alpha = 1, 2, 3 \) numbers the basis vectors. One of these basis vectors is naturally chosen to coincide with the axis of symmetry \( N \). Newton’s second law now gives (c.f. [6]), with the forces calculated in the previous section,

\[
m_B \frac{d}{dt} u_i = -\frac{1}{12} \mu \alpha \beta S b_1 (\delta_{ij} - 15N_{<i}N_{j>}) v_{<j,k>} N_k
\]
The time derivative of the angular momentum is obtained from Euler’s equations (see Goldstein [7])

\[ I_{ij} \frac{d\omega_j}{dt} + \epsilon_{ijk}\omega_jI_{kl}\omega_l = \frac{1}{2} \mu \alpha \beta N S^{3/2} b_2 \epsilon_{ijk} N_j v_{<k, l>N_l} \]

\[ + \left( \frac{2\pi k_B T}{m} \right)^{-1/2} \left\{ -3pS^{3/2} a_4 \epsilon_{ijk} N_j v_k - \frac{pS^2}{2} \left( a_3 \delta_{ij} + a_6 N_{<iN_j>} \right) \omega_j \right\} \]  

(21)

## STATIONARY SOLUTIONS

A first attempt to understand the equations of motion is provided by seeking a stationary solution. For this solution, we assume \( \omega = 0 \) and a constant velocity \( \mathbf{U} \). From the equations (20) and (21) we then get, using the values of \( a_1 \) and \( a_2 \) from the appendix,

\[ U_i = \frac{9\mu \beta N}{4p} \left( \frac{2\pi k_B T}{m} \right)^{1/2} \frac{\alpha \beta b_1}{4 + \frac{1}{2} \pi \alpha + [8 - (6 - \pi \alpha)] b_3} I N_i. \]  

(22)

In this expression, the purely geometrical parameter \( b_3 \) depends on the shape of the body: For a coin shaped body, or an extremely oblate body, \( b_3 = 1 \). For needle shape, or an extremely prolate body, \( b_3 = -\frac{1}{2} \). For a sphere, \( b_3 = 0 \). The definition of \( b_3 \) is

\[ b_3 = \frac{3}{2S} \int_S \left[ (\mathbf{n} \cdot \mathbf{N})^2 - \frac{1}{3} \right] dS. \]  

(23)

As mentioned earlier, \( b_1 \) is given by

\[ b_1 = \frac{1}{S} \int_S (\mathbf{n} \cdot \mathbf{N})^3 dS. \]  

(24)

We also recall that \( b_1 = 0 \) if the body is mirror symmetric in a plane orthogonal to the axis of symmetry.

Further, in (22), \( \mathbf{N} \) coincides with an eigenvector of traceless rate-of-deformation tensor \( (v_{<1,2>}) \), and \( l \) is the corresponding eigenvalue. For a velocity field with only one non-vanishing component, \( v_x(y) \), there are two eigenvectors with two non-vanishing eigenvalues of the corresponding traceless rate-of-deformation tensor. The first eigenvector is \( (e_x + e_y)/\sqrt{2} \), and has the eigenvalue \( l = \frac{1}{2} v_{x,y} \). The second eigenvector \( (-e_x + e_y)/\sqrt{2} \) has the corresponding eigenvalue \( l = -\frac{1}{2} v_{x,y} \).

A simple order of magnitude estimate of this velocity for the shearing discussed above is given by, if \( \lambda \) denotes the mean free path of the gas,

\[ |U| \sim \lambda v_{x,y}. \]  

(25)

## NON-DIMENSIONAL VARIABLES

To obtain numerical solutions to the equations of motion, we introduce a time-scale \( \tau \) according to

\[ \tau = \frac{m_B}{pS} \left( \frac{2\pi k_B T}{m} \right)^{1/2}. \]

This time scale is obtained from the force proportional to \( u \) in (20). For the flow field of the surrounding gas we consider a simple velocity field \( v_x(y) \) with only one non-vanishing component, \( v_{x,y} \). For the traceless rate-of-deformation tensor the only non-vanishing components then become
\[ v_{<x,y>} = v_{<y,x>} = \frac{1}{2} v_{x,y}. \] (26)

The corresponding eigendirections are the line \( x = y \) with the positive eigenvalue \( l = \frac{1}{2} v_{x,y} \), and the line \( x = -y \) with the negative eigenvalue \( l = -\frac{1}{2} v_{x,y} \).

To form a non-dimensional shearing, we divide it with the component (26). The eigenvalues of the resulting non-dimensional shearing are given by \( l_\pm = \pm 1 \). Further, we introduce a non-dimensional tensor of inertia according to \( I_{ij}^* = I_{ij}(m_B S)^{-1} \).

From the asymptotic velocity, we then get the velocity scale

\[ V = \frac{9 \mu \beta_N}{4p} \left( \frac{2\pi k_B T}{m} \right)^{1/2} \frac{1}{2} v_{x,y}. \]

The dimensionless variables \( t^*, v_i^*, x_i^*, \omega_i^* \) are now defined by

\[ t = \tau t^*, \quad u_i = V u_i^*, \quad x_i = V \tau x_i^*, \quad \omega_i = \tau^{-1} \omega_i^*. \] (27)

The non-dimensional version of the asymptotic velocity (22) then becomes, with the present scaling

\[ U_i^* = \frac{\alpha_x b_1}{4 + \frac{1}{2} \pi \alpha_x + [8 - (6 - \pi \alpha_x)]b_3} l^* N_i. \] (28)

**ORBITS OF A DOUBLE CONE.**

Here we investigate numerically the orbits of a special type of body, a 'double cone', shown in the figure below.

This body is axially symmetric, and is for the case \( s \neq \frac{1}{2} \) not mirror symmetric with respect to a plane orthogonal to the axis of symmetry.

This body is homogeneous and consists of two cones with a common base. The radius of the base is denoted by \( D \), and the total length by \( R \). The base is situated a distance \( s \cdot R \) from the left cusp, where the dimensionless parameter obeys \( 0 \leq s \leq \frac{1}{2} \). When \( s = 0 \) the double cone degenerates to a single cone with its cusp pointing in the direction of \( N \). When \( s = 1 \) we recover another single cone, pointing in the direction of \( -N \). We choose \( s \in [0, \frac{1}{2}] \), which means that \( N \) always points in the direction of the sharpest cusp of the double cone, see figure above. It should also be pointed out that in the expression for the final velocity (22), when \( s \in [0, \frac{1}{2}] \), is proportional to \( -l N_i \) for the double cones investigated.

The figures show the orbits of two different double cones, both with \( s = \frac{1}{4} \), in a gas subject to the simple shearing given by (26). We have here chosen \( \alpha_x = \frac{1}{8} \).

In the left figure, orbits of a blunt \((D/R = 2/5)\) double cone are plotted. In the left figure, orbits of a more slender body \((D/R = 1/4)\) are shown. The orbits are planar and lie in the \( x,y \)-plane. In each figure, 10 different orbits are represented. They all start at \( t^* = 0 \) in \((x,y) = (0,0)\). The initial angle between \( N \) and the horizontal axis is varied between 0 and \( 2\pi \). The initial velocity is parallel to \( N \), and the initial speed \( |u^*_0| = 0.0025 \sim |U^*|/10 \). The initial angular velocity is zero.
FIGURE 1. In the left picture, the planar orbits of blunt bodies with \( D/R = 2/5 \) starting from \((x, y) = (0, 0)\) tend to line up with the line \( x = -y \). In the right picture, the orbits of more slender bodies, with \( D/R = 1/4 \), finally wind up along the line \( x = y \). \( \alpha_r = 1/2 \).

It is obvious from these figures that the asymptotic transport of the blunt double cones takes place parallel the line \( x = -y \), and for the more slender bodies parallel to \( x = y \). It is found that in the final orientation of the blunt double cones, the sharpest cusp is directed parallel to the final velocity, whereas for the slender double cones, the sharpest cusp is directed in the direction opposite to the final velocity. This is in agreement with the expression for the final velocity (22), when \( s \in [0, \frac{1}{2}] \). We also point out, that the bodies with orbits of the left figure travel in both directions along the line \( x = -y \). The same goes for the the body orbits along the line \( x = y \) in the right figure. This symmetry stems from the tensor character of the transport mechanism, the shearing.

CONCLUSIONS

It has been shown that a body small compared to the mean free path in a shearing gas is subject to a, to the best of the knowledge of the authors, previously unknown transport mechanism, the \textit{Shearing Phoresis}. This transport takes place along the directions of the eigenvectors of the symmetric and traceless part of the velocity gradient. The final velocity of a transported body depends on the body shape, and is given by

\[
U_i = \frac{9\mu \beta N}{4p} \left( \frac{2\pi k_B T}{m} \right)^{1/2} \frac{\alpha_r b_1}{4 + \frac{1}{2} \pi \alpha_r + [8 - (6 - \pi \alpha_r)]b_3} l N_i.
\]  

(29)

Here, \( N \) is one of the eigenvectors of the symmetrical and traceless part of the velocity gradient, and \( l \) is the corresponding eigenvalue. For the simple type of shearing discussed above, we have the first eigenvector \( N = (e_x + e_y)/\sqrt{2} \) with the eigenvalue \( l = v_{x,y}/2 \), and the second eigenvector \( N = (e_x - e_y)/\sqrt{2} \) with the corresponding eigenvalue \( l = -v_{x,y}/2 \).

For bodies mirror symmetric in a plane orthogonal to the axis of symmetry, such as spheres, ellipses and right circular cylinders, this velocity vanishes.

Numerical simulations of the equations of motion for a double cone have been made for the case with a one-component shearing, \( v_{x,y} \). These suggest that slender double cones travel along the line \( x = y \), with the sharpest cusp in the direction opposite to the final velocity, and that more blunt double cones are transported along \( x = -y \), with the sharpest cusp in the direction parallel to the velocity.
ACKNOWLEDGMENT

L. Söderholm wishes to acknowledge a valuable discussion with Prof. I. Goldhirsch. This work has been supported by the Swedish Research Council for Engineering Sciences.

APPENDIX

In the following table, the values of the coefficients $a_1 - a_9$ and $b_2$ are listed:

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$\frac{1}{6} (8 + \pi \alpha_r)$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$\frac{1}{4} [8 - (6 - \pi) \alpha_r] (3 J_1 - 1)$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$\frac{1}{12} {[8 + (\pi - 2) \alpha_r] J_2 - [8 - (6 - \pi) \alpha_r] J_3 }$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$\frac{1}{12} {[8 + (\pi - 2) \alpha_r] J_2 - [8 - (6 - \pi) \alpha_r] J_3 }$</td>
</tr>
<tr>
<td>$a_5$</td>
<td>$\frac{1}{12} {[8 + (\pi - 2) \alpha_r] J_2 - [8 - (6 - \pi) \alpha_r] J_3 }$</td>
</tr>
<tr>
<td>$a_6$</td>
<td>$\frac{1}{8} {[16 - (10 - 2\pi) \alpha_r] J_5 - [16 - (12 - 2\pi) \alpha_r] J_6 }$</td>
</tr>
<tr>
<td></td>
<td>$- [24 - (12 - 3\pi) \alpha_r] J_7 - [24 - (18 - 3\pi) \alpha_r] J_8 }$</td>
</tr>
<tr>
<td>$b_2$</td>
<td>$J_{10} - J_{11}$</td>
</tr>
</tbody>
</table>

The integrals referred to in this table, $J_1 - J_9$, are given in the following table:

<table>
<thead>
<tr>
<th>Integrals</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_1$</td>
<td>$S^{-1} \int_S (N \cdot n)^2 , dS$</td>
</tr>
<tr>
<td>$J_2$</td>
<td>$S^{-3/2} \int_S x \cdot N , dS$</td>
</tr>
<tr>
<td>$J_3$</td>
<td>$S^{-3/2} \int_S (x \cdot n) (n \cdot N) , dS$</td>
</tr>
<tr>
<td>$J_4$</td>
<td>$S^{-3/2} \int_S (x \cdot N) (n \cdot N)^2 , dS$</td>
</tr>
<tr>
<td>$J_5$</td>
<td>$S^{-2} \int_S x^2 , dS$</td>
</tr>
<tr>
<td>$J_6$</td>
<td>$S^{-2} \int_S (x \cdot n)^2 , dS$</td>
</tr>
<tr>
<td>$J_7$</td>
<td>$S^{-2} \int_S (x \cdot N)^2 , dS$</td>
</tr>
<tr>
<td>$J_8$</td>
<td>$S^{-2} \int_S x^2 (n \cdot N)^2 , dS$</td>
</tr>
<tr>
<td>$J_9$</td>
<td>$S^{-2} \int_S (x \cdot n) (x \cdot N) (n \cdot N) , dS$</td>
</tr>
<tr>
<td>$J_{10}$</td>
<td>$S^{-3/2} \int_S (x \cdot N) (n \cdot N) , dS$</td>
</tr>
<tr>
<td>$J_{11}$</td>
<td>$S^{-3/2} \int_S (x \cdot n) (n \cdot N)^2 , dS$</td>
</tr>
</tbody>
</table>

The tensor of inertia, $I_{ij}$, contains two non-dimensional moments of inertia $I_1$ and $I_2$ become

$$I_1 = \frac{2}{3m_B S} \int_V \rho(x) x^2 \, d^3 x, \quad I_2 = \frac{1}{2m_B S} \int_V \rho(x) \left[ x^2 - 3(N \cdot x)^2 \right] \, d^3 x.$$

REFERENCES

7. Goldstein, R. Classical Mechanics, 2nd ed. (Addison-Wesley publishing company, New York, USA, 1980)