Comprehensive Kinetic Model For Weakly Rarefied Gases

Hakuro Oguchi

Institute of Space and Astronautical Science
Sagamihara C., Kanagawa-Ken ,Japan

Abstract. This paper concerns with the construction of a kinetic model equation which closely resembles the Boltzmann collision integrals for weakly rarefied gases of the monatomic molecules obeying the inverse power-law potentials. The collision integrals are dealt with under the assumption that the velocity distribution functions involved take the form of the thirteen-moment expansion. Through analytic manipulations we shall derive a comprehensive kinetic model equation which is applicable to analyses of the molecular motions under a wide range of positive-power interaction. The present formulation of the kinetic model equation suggests further possible extension to cases of the molecules obeying much practical potential laws.

INTRODUCTION

As is well known, the direct solution of the full Boltzmann equation is still difficult even by means of the recent advent of the computational technology. Apparently this is mainly due to the complexity of the collision integrals. So far there have been evolved a large variety of approximate approaches, one of which is to apply the kinetic model equation instead of the full Boltzmann equation. In the preceding paper, we presented a kinetic model equation, in which the velocity distribution function involved in the gain term of the collision integral was assumed to take an expression based upon the thirteen-moment approximation proposed by Grad. Then the gain term of the collision integral was obtained in an analytic formula, so that the kinetic model thus derived was of a form similar to the one of the existing model equations. In this paper, however the total collision integral is dealt with under the same approximation as the previous one for the velocity distribution functions. Furthermore, we shall much straightforwardly derive a kinetic model equation, which is comprehensively applicable to the cases of molecular gases obeying a wide range of the inverse power-law potentials and also may provide its possible extension for cases of other physical potentials. As an additional remark, the relation of the present kinetic model with the thirteen-moment system of equations is examined.

BOLTZMANN EQUATION WITH APPROXIMATE COLLISION TERM

Let us consider simple dilute monatomic gases which are free of any external force. As is well known, the state of gases is described by the full Boltzmann equation written with the familiar symbols, as follows:

\[
\frac{D F(t, \chi, \mathbf{v})}{Dt} = \int_{-\infty}^{\infty} d \mathbf{v}_1 \int_{0}^{\pi} d \chi \int_{0}^{2\pi} d \epsilon g_1 \sigma \sin \chi [F(\mathbf{v}_1^*)F(\mathbf{v}^*) - F(\mathbf{v}_1)F(\mathbf{v}^*)]
\]

where \(F\) is the velocity distribution function, \(\mathbf{v}\) the particle velocity, \(\sigma\) is the differential collision cross section, \(\chi\) the deflection angle in molecular collisions, \(\epsilon\) the azimuth, and \(g_1\) the relative speed given by

\[g_1 = \left| \mathbf{V} - \mathbf{V}_1 \right| = \left| \mathbf{v}^* - \mathbf{v}_1^* \right| \]

where \(\mathbf{V}\) is the peculiar (or thermal) velocity related to the mass velocity \(\mathbf{u}\) by \(\mathbf{V} = \mathbf{v} - \mathbf{u}\). For a specified pair of the pre-collision molecules with the velocities \(\mathbf{V}\) and \(\mathbf{V}_1\), the inverse collisions between pairs of molecules having the velocities \(\mathbf{V}^*\) and \(\mathbf{V}_1^*\) contribute to the gain of the number of \(\mathbf{V}\) molecules. It means that the double integrals with respect to \(\chi\) and \(\epsilon\) must be carried out over the surface of a sphere of the radius \(r = g_1/2\) having its center at the
center-of-mass velocity in the velocity space. The center-of-mass velocity $G_i$ for a pair of pre-collision molecules $(V, V_i)$ as well as pairs of the allowable inverse collision molecules $(V^*, V_i^*)$ are given, from the conservation of momentum, by

$$G_i = (V + V_i)/2 = (V^* + V_i^*)/2$$

With a half of the relative velocity $r = \langle g/2 \rangle$, we have the following relations from the elementary collision kinetics

$$V = r_i + G_i, \quad V_i = -r_i + G_i, \quad V^* = r + G_i, \quad V_i^* = -r + G_i$$

The above equations are derived from the energy conservation in molecular collisions. As will be noted later, in order to ensure the conservation of the lower-order five moments (mass, momentum, and energy), the set of the above relations must hold for the variables pertinent to the peculiar velocities. In this paper, we mainly concerns with the molecules obeying the inverse power – law interaction potential $U$; i.e.

$$U = K / \{(s-1)r_T^{s-1}\}$$

where $r_T$ is the distance of the reduced mass molecule from the fixed scattering center, $K$ is the potential constant and $(s-1)$ the power of the potential.

In the preceding paper, only the gain term in the collision integral was dealt with in somewhat schematic way in the discreet velocity space. In the present paper, however the total collision integral of the Boltzmann equation (1) is dealt with much analytically and straightforwardly, though the same approximation that in the preceding one for the velocity distribution function is made. For weakly rarefied gases of our primary concern, the velocity distribution functions involved in the collision term are assumed to take the form of the thirteen-moment approximation, primarily proposed by Grad. $U = K / \{(s-1)r_T^{s-1}\}$

$$F(V) = F_0(V)[1 + \frac{p_{mn}}{2RT}V_mV_n - \frac{q_m}{5RT}V_m(5 - \frac{V^2}{RT})]$$

where $V$ is the peculiar velocity of the molecules and $F_0$ is the local equilibrium distribution function given by

$$F_0(V) = (2\pi RT)^{-3/2} n \exp(-V^2/2RT)$$

In Eqs. (3) and (4), $n$ denotes the number density, $T$ the temperature, $p$ the pressure, $p_{mn}$ the non-divergence tensor pertinent to the stress tensor, $q_m$ the heat flux vector, $R$ the gas constant. Here we have the following relation between the non-divergence tensor $p_{mn}$ and the stress tensor $p_{mn}$; that is,

$$p_{mn} = p_{mn} - p \delta_{mn}$$

where $\delta_{mn}$ is the Kronecker delta. The subscript $m$ or $n$ denotes an axis among the Cartesian coordinates ($x_1, x_2, x_3$) adopted in the physical space concerned. The summation convention is adopted for the repeated subscripts; i.e.,

$$p_{mn}V_mV_n \equiv \sum_{m=1}^{3} \sum_{n=1}^{3} p_{mn}V_mV_n, \quad q_mV_m \equiv \sum_{m=1}^{3} q_mV_m$$

From the definition of the stress tensor $p_{mn}$ we have

$$p_{mn} \equiv \sum_{n=1}^{3} p_{mn} = \sum_{m=1}^{3} p_{mm} - 3p = 0$$

Thus the thirteen independent moments $n, u, T, p, p_{mn}, q_m$ are relevant to the velocity distribution function, together with the law of state $p = k n T$ where $k$ is the Boltzmann constant. In terms of the velocity distribution function $F$, the thirteen moments concerned are given as follows:

$$n = \int F \, dV, \quad u = \frac{1}{n} \int \mathbf{v} F dV, \quad \frac{3}{2} kT = \frac{1}{n} \int \frac{mV^2}{2} F dV,$$

$$p_{mn} = \int V_mV_n FdV - 3p \delta_{mn}, \quad q_m = \frac{m}{2} \int V^2 V_m F dV,$$

where the integrations are performed over a whole velocity space. Since, as mentioned before, we primarily concerned with the weakly rarefied gases, the velocity distribution functions involved in the collision term are assumed to take the form given by the thirteen- moment approximation (3). If this is allowed, the following relation is derived with neglect of the higher-order cross product terms,
\[
F(V_1^*)F(V_1^*) = F_0(V^*)F_0(V_1^*)[1 + \frac{p_{mn}}{2RT_p}(V_m^*V_1^* + V_{1m}^*V_1^*) - \frac{g_{mn}V_1^*}{5RT_p}(5 - \frac{V_1^{*2}}{RT}) - \frac{g_{m1}V_{1m}^*}{5RT_p}(5 - \frac{V_{1m}^{*2}}{RT})]
\]

With the aid of the variable relations (2), the above equation becomes in terms of kinetic model. First, the double integrals regarding \(x\) and \(\xi\) are generated by the rotation of the spatial coordinates \((x, y, z)\) to \((r, \theta, \phi)\) and \((\xi, \eta, \zeta)\) such that \(\xi_3\)-axis coincides with \(G_1\)-direction, the \(\xi_1\)-axis lies on the plane containing both \(x_3\) and \(\xi_3\)-axes, and the \(\xi_2\)-axis is normal to that plane. We note that the coordinates thus generated are pertinent to \(r_1\).

Quite similarly, we also have

\[
\frac{F(V^*)F(V_1^*)}{F_0(V^*)F_0(V_1^*)} = 1 + \frac{p_{mn}}{RT_p}(r_{1m}r_{1n} + G_{1,m}G_{1,n}) - \frac{2g_{mn}}{RT_p}\{G_{1,m} - G_{1,m}(r^2 + G^2) - \frac{2}{5RT_r}r_{m,n}G_{1,m}\}
\]

In view of the energy conservation in binary collisions of the molecules, we have

\[
F_0(V^*)F_0(V_1^*) = F_0(V)F_0(V_1) = n^2(2\pi RT)^{-3}\exp\left(-\frac{(V^2 + V_1^2)}{2RT}\right)
\]

The substitution of Eqs. (6) and (7) into Eq.(1) yields

\[
\frac{DF(t,x,v)}{Dt} = \int dV_1 F_0(V)F_0(V_1)(2\pi)\left[\frac{p_{mn}}{RT_p}\int_0^{2\pi} d\phi \sin\chi(r_{m,n} - r_{1,m,n}) + \right]
\]

\[
+ \frac{4g_{mn}G_{1,m}r_{m,n}}{5(2RT_p)}\int_0^{2\pi} d\phi \sin\chi(r_{m,n} - r_{1,m,n})\]

In what follows, we shall perform the above five-fold integrations to derive a closed analytic formula or our kinetic model. First, the double integrals regarding \(\chi\) and \(\epsilon\) in Eq. (8) are dealt with for a fixed pair of \(V\) and \(V_1\). Next, by performing the remaining ternary integrals regarding \(V_1\), the collision integrals in Eq. (8) are obtained in an analytic form which is computable for actual applications.

**COLLISION INTEGRALS FOR A FIXED PAIR OF \(V\)-AND \(V_1\) MOLECULES**

As regards the integrals on the right hand side of Eq. (8), we first deal with the integral regarding the azimuth \(\epsilon\) and then the integral regarding the deflection angle \(\chi\). We introduce the new Cartesian coordinates \((\xi_1, \xi_2, \xi_3)\) which are generated by the rotation of the spatial coordinates \((x_1, x_2, x_3)\) around the origin located at \(G_1\). Then the \(\xi_3\)-axis is chosen such that its direction coincides with \(r_1\)-direction, the \(\xi_1\)-axis lies on the plane containing both \(x_3\) and \(\xi_3\)-axes, and the \(\xi_2\)-axis is normal to that plane. We note that the coordinates thus generated are pertinent to \(r_1\).

If we choose the polar axis to coincide with the \(\xi_3\)-axis, then a half of the relative velocity \(r\) is expressed as \((r \sin\chi \cos\epsilon, r \sin\chi \sin\epsilon, r \cos\chi)\) component wise in the \(\xi\)-coordinates. Let us denote the angle between \(x_3\) and \(\xi_3\)-axes by \(\theta\) and the angle of the \(x_1\)-axis, making against the plane containing the both \(x_3\) and \(\xi_3\)-axes, by \(\psi\). Then, the \((\mathbf{x-\xi})\) transformation \(T_1\) is given by the following relation,

\[
\begin{pmatrix}
\frac{r_1}{r} \\
\frac{r_2}{r} \\
\frac{r_3}{r}
\end{pmatrix} = \begin{pmatrix}
\cos \psi \cos \phi & -\sin \phi & \sin \psi \cos \phi \\
\cos \psi \sin \phi & \cos \phi & \sin \psi \sin \phi \\
-\sin \theta & 0 & \cos \theta
\end{pmatrix} \begin{pmatrix}
r \sin \chi \cos \epsilon \\
0 \\
r \cos \chi
\end{pmatrix}
\]

Since we have

\[
\begin{pmatrix}
\frac{r_{11}}{r} \\
\frac{r_{12}}{r} \\
\frac{r_{13}}{r}
\end{pmatrix} = \begin{pmatrix}
r \sin \theta \cos \psi & r \sin \theta \sin \psi & r \cos \psi
\end{pmatrix}
\]

Eq. (9) is also rewritten as

\[
\begin{pmatrix}
\frac{r_1}{r} \\
\frac{r_2}{r} \\
\frac{r_3}{r}
\end{pmatrix} = \begin{pmatrix}
\frac{r_{11}r_{13} - r_{12}r_{12}^*}{r_{13}} & \frac{r_{12}r_{12}^* - r_{13}r_{13}^*}{r_{12}} & \frac{r_{13}r_{13}^* - r_{12}r_{12}^*}{r_{13}} \\
\frac{r_{12}r_{12}^* - r_{13}r_{13}^*}{r_{12}} & \frac{r_{13}r_{13}^* - r_{12}r_{12}^*}{r_{13}} & \frac{r_{12}r_{12}^* - r_{13}r_{13}^*}{r_{12}} \\
-r_{13}r_{13}^* & 0 & r_{13}r_{13}^*
\end{pmatrix} \begin{pmatrix}
r \sin \chi \cos \epsilon \\
r \sin \chi \sin \epsilon \\
r \cos \chi
\end{pmatrix}
\]
where
\[ r^* = \left( r_{1,1}^2 + r_{1,2}^2 \right)^{1/2} \]
\[ r^2 = r_{1,m} r_{1,n} = r_m r_n \]

As regards the integrals involved in Eq. (8), the following lemma derived by the transformation of Eq. (10) is useful,
\[
\int_{0}^{2\pi} r_m r_n d\varepsilon = 2\pi r_{1,m} r_{1,n} \left( 1 - \frac{3}{2} \sin^2 \chi \right) + \delta_{mn} \pi r^2 \sin^2 \chi
\]  
(11)

Thus we obtain
\[
\int d\chi \sin \chi \int_{0}^{2\pi} d\varepsilon (r_m r_n - r_{1,m} r_{1,n}) = -\frac{3}{2} \sigma_m r_{1,m} r_{1,n} + \delta_{mn} \sigma_n r^2 \frac{2}{2}
\]  
(12)

The \( \sigma_m \) is the viscosity cross-section defined by
\[
\sigma_m = 2\pi \int_{0}^{\pi} \sin^3 \chi d\chi = A^*(s)(2r)^{-4/(s-1)}
\]
(13)

where
\[ A^* = 2\pi (2K/m)^{2/(s-1)} A_2(s) \]

The function \( A_2(s) \) is tabulated in the contexts for a variety of the potential exponent \( s \). By the use of Eq. (13) with the relation \( p_{mm} = 0 \), we have from Eq. (8)
\[
\frac{DF(t, x, \nu)}{Dt} = \int_{-\infty}^{\infty} dV_1 A^*(2r)^{(s-5)/(s-1)} F_0(V) F_0(V_1) \left[ -\frac{3}{2} p_{mm} r_{1,m} r_{1,n} + \frac{2q_m}{5p(\pi^2)^2} (r^2 G_{1,m} - 3r_{1,m} r_{1,n} G_{1,n}) \right]
\]
With the aid of the variable transformation (2) the above equation becomes
\[
\frac{DF(t, x, \nu)}{Dt} = \int_{-\infty}^{\infty} dV_1 A^* n^2 (2\pi)^{(s-5)/(s-1)} \exp[-(2r^2 - 2r_{1,m} V_m + V^2)/\pi]\]
\[ \times \left[ -\frac{3}{2} p_{mm} r_{1,m} r_{1,n} + \frac{2q_m}{5(\pi^2)^2} (r^2 V_m + 2r^2 r_{1,m} r_{1,n} V_n) \right]
\]
(14)

**DERIVATION OF KINETIC MODEL**

In order to conclude our kinetic modeling of the Boltzmann equation, we must perform the integration of Eq.(14). Since the integrand in Eq. (14) is expressed in terms of \( r \), the integral variable \( V_1 \) must be transformed into \( r \) for fixed \( V \) as
\[ dV_1 = -d\mathbf{g}_r = -d(2r) \]

Introducing the polar coordinates \( (r, \kappa, \omega) \) having the \( \xi_3 \) -axis in \( V \)-direction as the polar axis and \( V \) as its origin, then we have
\[
\int_{-\infty}^{\infty} dV_1 \left[ \right] = -8\pi \int_{0}^{\infty} r^2 dr \int_{0}^{\pi} \sin \kappa d\kappa \int_{0}^{2\pi} d\omega \left[ \right]
\]

In what follows, without any confusion we employ the non-dimensional variables defined by
\[ r = r/(RT)^{1/2}, \quad r_m = r_{1,m}/(RT)^{1/2}, \quad C = V/(RT)^{1/2}, \quad C_m = V_m/(RT)^{1/2} \]

Then Eq. (14) is rewritten as
\[
\frac{DF}{Dt} = \frac{A^* n^2 (4\pi)^{(s-5)/2(s-1)}}{\pi^3 (2\pi)^{3/2}} \exp(-C^2) \int_{0}^{\infty} \int_{0}^{\pi} \int_{0}^{2\pi} \sin \kappa d\kappa d\omega \times
\]
\[ \times r^{2(s-5)/(s-1)} \exp(-2r^2 + 2r_m C_m) \left[ \frac{3}{2} p_{mm} r_{1,m} r_{1,n} - \frac{2q_m}{5p(\pi^2)^{1/2}} (r^2 C_m + 2r^2 r_m - 3r_{1,m} r_{1,n} C_n) \right]
\]
(16)
In Eq. (15) the variables $r$ and $r_{1,m}$ are pertinent to $V_1$ for a fixed $V$. In order to perform the integration, we introduce the similar transformation to $T_1$ of Eq. (9). In this case the $\xi_3$ is chosen in $V$-direction, and the $\xi_1$-axis chosen to lie on the $x_2 - \xi_3$ plane. The angle of $\xi_3$ against $x_3$ is denoted by $\theta$, and the angle of $\xi_1$-axis making for the $x_2 - \xi_3$ plane is denoted by $\phi$. Quite similarly to the derivation of the transformation $T_1$, we have the following transformation $T_2$ expressed in terms of non-dimensional variables $r$ and $r_{1,m}$.

$$
\begin{pmatrix}
r_1 \\
r_2 \\
r_3
\end{pmatrix} =
\begin{pmatrix}
\cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\
\cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\
-\sin \theta & 0 & \cos \theta
\end{pmatrix}
\begin{pmatrix}
r \sin \kappa \cos \omega \\
r \sin \kappa \sin \omega \\
r \cos \kappa
\end{pmatrix}
$$

where

$$(V_1/V, V_2/V, V_3/V) = (C_1/C, C_2/C, C_3/C) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

In performing the integration in Eq. (16), we use the following relations derived by use of Eq. (17)

$$
\int_0^{2\pi} r_m C_m d\omega = 2\pi \frac{C_m}{C} \int_0^{2\pi} r \cos \kappa d\omega 
$$

$$
\int_0^{2\pi} r_m r_n^\prime \omega d\omega = 2\pi \frac{C_m}{C} C_n \int_0^{2\pi} r^2(1 - \frac{3}{2} \sin^2 \kappa) + \delta_{mn} \pi r^2 \sin^2 \kappa d\omega
$$

It should be noted that the last equation is quite similar to Eq. (11). The resulting formula after integration of Eq. (16) is expressed as follows:

$$
\frac{DF(t, x, \nu)}{Dt} = A^* n^2 (4RT)^{(s-5)/(s-1)} \pi^2 (RT)^{3/2} \exp(-C^2)\left[-\frac{3p_{mn}}{2p} \frac{C_m C_n}{C^2} P - \frac{4q_m}{5p(RT)^{1/2}} C_n (P - \frac{2}{C} Q)\right]
$$

where $P$ and $Q$ are the functions of $C$ alone given by

$$
P(C) = \int_0^\infty r^{2(s-3)/(s-1)} \exp(-2r^2)\left[\left(\frac{r^2}{2C} - \frac{3r}{2C^2} + \frac{3}{4C^3}\right)\exp(2Cr) - \left(\frac{r^2}{C} + \frac{3r}{2C^2} + \frac{3}{4C^3}\right)\exp(-2Cr)\right] dr
$$

$$
Q(C) = \int_0^\infty r^{2(s-3)/(s-1)} \exp(-2r^2)\left[\left(\frac{r^3}{2C} - \frac{r^2}{4C^2}\right)\exp(2Cr) + \left(\frac{r^3}{2C} + \frac{r^2}{4C^2}\right)\exp(-2Cr)\right] dr
$$

Equation (18) with $P$ and $Q$, respectively, given by Eqs. (19) and (20) is the kinetic model equation which contains the thirteen moments of the velocity distribution function. The parameter involved, $A^*$ or $A_2$, pertinent to the viscosity cross-section $\sigma_\mu$ is presumed to be related to the viscosity given by

$$
\mu = \frac{5m(RT/\pi)^{1/2}(2mRT/K)^{2/(s-1)}}{8A_2(s)\Gamma\{4 - 2/(s - 1)\}}, \quad \Gamma(s) = \int_0^\infty \exp(-x) dx
$$

where $\Gamma(s)$ is the gamma function. From Eqs. (13) and (21) we derive the following relation

$$
A^* n = \frac{5\sqrt{\pi}}{\Gamma\{4 - 2/(s - 1)\}} \frac{2^{2(s-3)/(s-1)} P}{\Gamma\{4 - 2/(s - 1)\} (RT)^{(s-5)/(s-1)} \mu}
$$

Apparently the present kinetic model constitutes a closed system for computation. It should be noted that the model equation (18) contains the measurable thirteen moments alone and the right hand side pertinent to the collision processes does no differentials explicitly.
Special Molecular Models

Maxwell Molecules

For Maxwell molecules ($s=5$) the integrals (19) and (20) simply reduce to the followings

$$P = \frac{(2\pi)^{1/2}}{16} C^2 \exp\left(\frac{C^2}{2}\right) \quad Q = \frac{(2\pi)^{1/2}}{64} C(5 + C^2) \exp\left(\frac{C^2}{2}\right)$$

Consequently, we have the kinetic model equation for the Maxwell molecules as follows:

$$\frac{DF(t, x, v)}{Dt} = \frac{A^* n^2}{(2\pi RT)^{3/2}} \exp\left(-\frac{V^2}{2RT}\right) \left[-P_1 \frac{p_{mn} V_m V_n}{p RT} - Q_1 \frac{q_m V_m}{pRT}\right]$$

where from Eq. (22) we have

$$A^* n = \left(\frac{4}{3}\right) \frac{p}{\mu}$$

which represents the collision frequency. We note the present model is qualitatively similar to the model proposed by Shakov except that his model remains an indefinite parameter.

Hard Sphere Molecules

For hard sphere molecules ($s \to \infty$) we also obtain the following kinetic model equation,

$$\frac{DF(t, x, v)}{Dt} = \frac{A^* n^2 (RT)^{1/2}}{(2\pi RT)^{3/2}} \exp\left(-\frac{V^2}{2RT}\right) \left[-P_1 \frac{p_{mn} V_m V_n}{p RT} - Q_1 \frac{q_m V_m}{pRT}\right]$$

where

$$P_1 = \frac{3}{4(2\pi)^{1/2}} \left(1 + \frac{2}{C^2} - \frac{3}{C^4}\right) \exp\left(-\frac{C^2}{2}\right) + \frac{3}{8} \left(C^3 - \frac{3}{C^3} + \frac{3}{C^5}\right) \text{erf}\left(\frac{C}{\sqrt{2}}\right)$$

$$Q_1 = \frac{1}{5(2\pi)^{1/2}} \left(C^2 - 4 - \frac{3}{C^2}\right) \exp\left(-\frac{C^2}{2}\right) + \frac{1}{10} \left(C^3 - \frac{3}{C^3} - \frac{15}{C^5}\right) \text{erf}\left(\frac{C}{\sqrt{2}}\right)$$

with the definition of the error function:

$$\text{erf}(a) = \frac{2}{\sqrt{\pi}} \int_{0}^{a} \exp(-t^2) dt$$

The parameter $A^*$ is given from Eq.(22) as follows:

$$A^* n = \frac{5\sqrt{\pi}}{24} \frac{p}{\mu} (RT)^{-1/2}$$

Another Specific Molecules ($s=3$)

For another specific molecules ($s=3$), we have a closed analytic expression for the collision integral; i.e.

$$\frac{DF(t, x, v)}{Dt} = \frac{A^* n^2 (RT)^{-1/2}}{(2\pi RT)^{3/2}} \exp\left(-\frac{C^2}{2}\right) \left[-P_3 \frac{p_{mn} V_m V_n}{p RT} - Q_3 \frac{q_m V_m}{pRT}\right]$$

where

$$P_3 = \frac{3}{4(2\pi)^{1/2}} \left(\frac{1}{C^2} - \frac{3}{C^4}\right) \exp\left(-\frac{C^2}{2}\right) + \frac{3}{8} \left(\frac{1}{C^3} - \frac{2}{C^5}\right) \text{erf}\left(\frac{C}{\sqrt{2}}\right)$$

$$Q_3 = \frac{1}{10(2\pi)^{1/2}} \left(C^2 - 4 - \frac{3}{C^2}\right) \exp\left(-\frac{C^2}{2}\right) + \frac{1}{10} \left(C^3 - \frac{3}{C^3} - \frac{15}{C^5}\right) \text{erf}\left(\frac{C}{\sqrt{2}}\right)$$
\[ Q_3 = \frac{1}{5(2\pi)^{1/2}} (1 - \frac{7}{C^2}) \exp(-\frac{C^2}{2}) + \frac{1}{10} (C - \frac{6}{C} + \frac{7}{C^2}) \text{erf}(\frac{C}{\sqrt{2}}) \]

and

\[ A^* n = \frac{5\sqrt{\pi}}{2} \frac{p}{\mu} (RT)^{1/2} \]

As shown above, the analytic expressions of the collision term have been derived for the specific molecular models with the potential power \( s = 3, 5 \) (Maxwell molecules) and infinity (hard sphere molecules) (see Eqs. (23-25)).

**Relation With Thirteen –Moment System of Equations**

Referring to the context by Kogan, we have the following moment equations which are similar to the one primarily derived by Grad (so-called thirteen-moment system of equations),

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r} (\rho u_r) &= 0 \\
\frac{\partial u_i}{\partial t} + u_r \frac{\partial u_i}{\partial x_r} + \frac{1}{\rho} \frac{\partial P_{\mu i}}{\partial x_r} &= \frac{1}{\rho} \int_V F_{\text{col}} \, dV \equiv I_1 \\
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r} (u_i, p) + \frac{2}{3} P_{\mu} \frac{\partial u_i}{\partial x_r} + 2 \frac{\partial q_L}{\partial x_r} &= 0 \\
\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_r} (u_i, p_j) + \left[ \frac{2}{5} \frac{\partial q_{ij}}{\partial x_j} - \frac{2}{3} \frac{\partial q_{ij}}{\partial x_r} + p_{ij} \frac{\partial u_i}{\partial x_r} + p_{jr} \frac{\partial u_j}{\partial x_r} + \frac{2}{3} \frac{\partial q_{ij}}{\partial x_r} \right] &= \frac{m}{\rho} \int_V F_{\text{col}} \, dV \equiv I_2 \\
\frac{\partial q_{ij}}{\partial t} + \frac{\partial}{\partial x_r} (u_i, q_j) + \frac{7}{5} q_r \frac{\partial q_{ij}}{\partial x_j} + \frac{2}{5} q_r \frac{\partial u_i}{\partial x_j} + \frac{2}{5} q_r \frac{\partial u_i}{\partial x_r} + \frac{kT}{m} \frac{\partial p_{ij}}{\partial x_r} + \frac{7}{2} p \frac{\partial q_{ij}}{\partial x_r} - \frac{p_{ij}}{\rho} \frac{\partial P_{ij}}{\partial x_r} &= \frac{2}{m} \int_V V^2 F_{\text{col}} \, dV \equiv I_3 \\
\end{align*}
\]

where \( F_{\text{col}} \) denotes the collision term given by the right hand side of Eq. (18). The collision integrals \( I_s \) for the special molecular models mentioned before are evaluated. First, it is analytically confirmed that \( I_1 = 0 \). For any other molecular models of positive \( s \)-potential it was also confirmed numerically. Consequently, for the present kinetic model the conservation law pertinent to the basic lower moments was ensured. For the remaining integrals, we have

\[
I_2 = m \int_{-\infty}^{\infty} V_i V_j F_{\text{col}} \, dV, \quad I_3 = \frac{m}{2} \int_{-\infty}^{\infty} V_i V^2 F_{\text{col}} \, dV
\]

After some analytical manipulations, for all of three specific molecular models mentioned above, we obtain

\[
I_2 = -\frac{p}{\mu} p_{ij}, \quad I_3 = -\frac{2}{3} \frac{p}{\mu} q_i
\]

The above relations are also numerically ensured for several power potential of positive \( s \), within the accuracy of tolerance. Thus, the system of moment equations is identified with the one primarily proposed by Grad. As the results, the consistency of the presumption of Eq. (21) is also confirmed.
CONCLUDING REMARKS

We proposed the kinetic model equation, in which the collision integral is given by a closed analytical formula in terms of the thirteen moments of the velocity distribution function. In view of the underlying approximation, this model may be applicable to analyses of the weakly rarefied gases flows of the monatomic molecules obeying the inverse power-law potentials. Actual computational analyses on rarefied gas flows will be made based upon the kinetic model together with the recent evolution of computational techniques (for example, references 8, 9). Moreover, further extension of the present model may be expected for the molecules obeying such interaction potentials that the viscosity cross-section \( \sigma_{\mu} \) is explicitly specified as a function of the relative speed \( g (=2r) \) in the form of Eq. (13).

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REFERENCES


APPENDIX

According to Chapman-Enskog procedure, we can derive the well-known relations for the viscosity as well as heat conduction of gases from Eqs (26) and (27) along with Eq. (28) as follows. Namely, we have the lower–order relations by use of Eqs. (26), (27) and (28) as follows:

\[
\mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) = -p_y, \quad \frac{15}{4} \mu \frac{\partial RT}{\partial x_j} = -q_i
\]

From the second relation, the coefficient of heat conduction \( \lambda \) is related to the coefficient of viscosity \( \mu \), by

\[
\lambda = (15/4) R \mu
\]

and thus we have the correct Prandtl number \( Pr = 2/3 \).