A Variable Multigroup Approach to the Nonlinear Boltzmann Equation Based on the Method of Weighted Residuals

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Abstract. A multigroup approximation to the Boltzmann equation with an arbitrary partition of the speed variable combined with a spherical harmonics expansion of the directional dependence of the distribution function is presented. Based on the method of weighted residuals, weight functions assure the exact conservation of particle density, momentum and energy density. A numerical example shows the applicability of this very flexible method to study the relaxation of a homogeneous gas far from equilibrium.

INTRODUCTION

The Boltzmann equation constitutes a very powerful tool for the description of rarefied gases [1]. However, its complicated structure as a nonlinear integro-differential equation prevents a rigorous mathematical solution. Consequently, there have been several attempts to construct approximation methods, so-called model Boltzmann equations. These model equations allow one to obtain suitable information about the properties of rarefied gases thus avoiding many mathematical and numerical difficulties.

One of these approaches is the concept of generating moment equations. This method is based on forming moments of the Boltzmann equation in connection with an appropriate closure of the resulting set of equations. Here, Grad’s 13 moments method should be mentioned [2]. Since every closure method includes some intuitive assumptions, the concept of moment equations has been extended in a way that it immediately leads to a closed set of equations. Such a generalization can be performed by means of a discretization of the modulus of the speed variable, which ends in the well-known multigroup approaches.

Multigroup approaches are very common and successful ways to treat the linear Boltzmann equation. As a consequence, there have been many attempts to transmit this formalism to the nonlinear Boltzmann equation as well. In a recent paper [3], a multigroup approximation to the nonlinear Boltzmann equation was derived and its main features were discussed. It has been shown that this approach guarantees particle conservation exactly, while momentum and energy conservation are only approximately fulfilled. On the other hand, a conservative two group model for the nonlinear Boltzmann equation has already been established [4]. This model fulfills all the required conservation properties, provided that the underlying transition probabilities obey certain conditions.

Another recently published multigroup model called revised multigroup approach [5] for the nonlinear Boltzmann equation is free of the restrictions of the other models. This approximation, based on the method of weighted residuals [8], leads to a set of integro-differential equations, which ensures exactly the conservation of particle density, momentum and energy density without any ad hoc assumptions about the scattering process. It is the aim of this paper to extend this conservative multigroup approach by a method similar to Grad’s method. Thus, the $P_2$ approximation is used for treating the angular dependence of the distribution function. Since this approximation is performed in each speed interval, an essential improvement of the gained results in comparison to those of Grad’s method is obtained. This procedure leads to a set of partial differential equations. In other words, this method results in a description of the physical state of a gas in each speed interval, which corresponds with an extended thermodynamics approach [6].
THE REVISED MULTIGROUP APPROACH

In the kinetic theory, the temporal and spatial evolution of the distribution function \( f = f(x, v, t) \) of monatomic rarefied gases is governed by the Boltzmann equation. It reads in the scattering kernel formulation [7] after the polar decomposition of the velocity variable \( v = v\hat{\Omega} \) with \( v = |v| \) and \( \hat{\Omega} = v/v \) and the introduction of the new function \( F(x, v, \hat{\Omega}, t) := v^2 f(x, v, t) \):

\[
\frac{\partial F}{\partial t} + v\hat{\Omega} \cdot \nabla F = J[F, F]
\]  

(1)

with the collision term

\[
J[F, F] = \int_0^\infty dv' \int_0^\infty dv'' \int_S^2 d\hat{\Omega}' \int_S^2 d\hat{\Omega}'' g(|v'\hat{\Omega}' - v''\hat{\Omega}''|) \Pi(v'\hat{\Omega}', v''\hat{\Omega}'' \rightarrow v\hat{\Omega}) F(v', \hat{\Omega}') F(v'', \hat{\Omega}'')
\]

(2)

Here, \( g \) denotes the collision frequency, while \( \Pi \) is the scattering kernel. In the case of isotropic, elastic scattering, \( g \) and \( \Pi \) are given by

\[
g(|v\hat{\Omega} - v'\hat{\Omega}'|) = |v\hat{\Omega} - v'\hat{\Omega}'| \sigma(|v\hat{\Omega} - v'\hat{\Omega}'|), \quad \Pi(v'\hat{\Omega}', v''\hat{\Omega}'' \rightarrow v\hat{\Omega}) = \frac{v^2 \delta((v\hat{\Omega} - v'\hat{\Omega}') \cdot (v\hat{\Omega} - v''\hat{\Omega}''))}{\pi |v'\hat{\Omega}' - v''\hat{\Omega}''|}
\]

(3)

with the total scattering cross section \( \sigma \).

The first step to construct the revised multigroup approximation consists of introducing a partition of the non-negative real axis \( v \in [0, \infty) \) in \( N \) intervals located between \( N + 1 \) speed-knots \( w_i, i = 0, 1, 2, \ldots, N \). Here, \( w_N \) has to be taken high enough such that \( F(w_N, \hat{\Omega}) \) vanishes. Moreover, \( w_0 \) is set to zero.

In the second step, we represent the particle distribution function as a finite series,

\[
F(x, v, \hat{\Omega}, t) = \sum_{i=1}^N \psi_i(x, \hat{\Omega}, t) \psi_i(v),
\]

(4)

with \( N \) shape functions \( \psi_i(v) \) and \( N \) new unknown functions \( \psi_i \) which are independent of the modulus of velocity. Moreover, we demand that the shape functions \( \psi_i(v), i = 1, 2, \ldots, N \) fulfill the following relations

\[
(i) \quad \psi_i(v) \geq 0, \quad v \in [0, \infty), \\
(ii) \quad \psi_i(v) \equiv 0, \quad v \notin [w_{i-1}, w_i], \\
(iii) \quad \int_{w_{i-1}}^{w_i} dv \psi_i(v) = 1,
\]

(5)

which allows us to obtain the new unknowns \( \psi_i \) by the simple integration

\[
\psi_i(x, \hat{\Omega}, t) = \int_{w_{i-1}}^{w_i} dv F(x, v, \hat{\Omega}, t).
\]

(6)

As suggested by the method of weighted residuals [8], a set of approximative differential equations can be constructed in the following way: the ansatz (4) for the distribution function is inserted into the Boltzmann equation and the result, weighted with \( L \) different weight functions \( \lambda_l \), is integrated over \( N/L \) parts of the whole variation range of the independent variable \( v \). This procedure leads to a closed set of conditional equations for the new unknowns \( \psi_i \).

Macroscopic quantities, as for example the particle density, the drift velocity or the energy density are power moments of the distribution function. Since we want to derive multigroup model Boltzmann equations obeying the exact conservation laws for mass, momentum and energy, we choose powers of \( v \) as suitable weight functions, \( \lambda_l(v) = v^l, l = 0, 1, \ldots, L - 1 \).

Moreover, we have to specify the intervals, we integrate over. For constructing \( N \) equations with \( L \) different weight functions \( \lambda_l \), \( N/L \) intervals are necessary. They are determined by joining \( L \) adjacent intervals \( I_i = [w_{i-1}, w_i], i = 1, 2, \ldots, N \) to \( N/L \) macro-intervals \( I_q^N = [w_{(q-1)\cdot L}, w_{q\cdot L}], q = 1, 2, \ldots, N/L \). Figure 1 displays a possible partition of the speed axis for \( N = 12 \), the macro-intervals \( I_q^N \) for \( L = 3 \) as well as a possible shape function \( \psi_6 \).
The application of the above considerations to the Boltzmann equation (1) in connection with (4) leads to a set of \( N \) integro-differential equations. They are called the revised multigroup approach (RMA):

\[
\sum_{i=I(L(q-1)+1)}^{L_q} \left( \langle v^i \rangle_t \frac{\partial \varphi_i}{\partial t} + \langle v^{i+1} \rangle_t \tilde{\Omega} \cdot \nabla \varphi_i \right) = \sum_{i=I(L(q-1)+1)}^{L_q} \left( - \sum_{j=1}^{N} \int_{S^2} d\tilde{\Omega}' \ g_{ij}^{(l)} (\tilde{\Omega}' \cdot \tilde{\Omega} ) \varphi_j (\tilde{\Omega}) \varphi_i (\tilde{\Omega}') \right) + \sum_{j=1}^{N} \int_{S^2} d\tilde{\Omega}' \ \int_{S^2} d\tilde{\Omega}'' \ g_{jj}^{(l)} (\tilde{\Omega}' \cdot \tilde{\Omega}'' ) \Pi_{jj}^{(l)} (\tilde{\Omega}' , \tilde{\Omega}'' \rightarrow \tilde{\Omega}) \varphi_j (\tilde{\Omega}') \varphi_j (\tilde{\Omega}'') \right) \tag{7}
\]

with the abbreviations

\[
\langle v^i \rangle_t = \int_{w_{i-1}}^{w_i} dv^i \psi_i(v),
\]

\[
g_{ij}^{(l)} (\tilde{\Omega} \cdot \tilde{\Omega}') = \int_{w_{i-1}}^{w_i} dv^i \psi_i(v) \int_{w_{j-1}}^{w_j} dv^j \psi_j(v) g(|v\tilde{\Omega} - v'\tilde{\Omega}'|),
\]

\[
\Pi_{jk}^{(l)} (\tilde{\Omega} , \tilde{\Omega}' \rightarrow \tilde{\Omega}) = \frac{1}{g_{jk}^{(l)} (\tilde{\Omega} \cdot \tilde{\Omega}')} \int_{w_{i-1}}^{w_i} dv^i \int_{w_{j-1}}^{w_j} dv^j \psi_j(v) \int_{w_{k-1}}^{w_k} dv^k \psi_k(v) g(|v'\tilde{\Omega}' - v''\tilde{\Omega}''|) \Pi (v'\tilde{\Omega}' , v''\tilde{\Omega}'' \rightarrow v\tilde{\Omega}).
\]

As shown below, \( L \geq 3 \) is a necessary condition for the exact fulfillment of conservation of mass, momentum and energy. This fact constitutes the lower limit for \( L \). On the other hand, \( L > 3 \) is combined with the emerging of high order moments in the multigroup approach, which have a physical meaning, but does not lead to quantities which are conserved in binary collisions. While this aspect only leads to interpretational problems, a second consideration explains more clearly, why \( L = 3 \) is certainly a suitable choice. A microscopic quantity \( g_1(v)g_2(\tilde{\Omega}) \) is related to a macroscopic one, \( G \), by forming moments, i.e. \( g_1(v)g_2(\tilde{\Omega}) \) is weighted with the distribution function \( F \) and integrated with respect to \( v \). Moreover, a quantity \( G_T \) to the microscopic quantity \( g_1(v)g_2(\tilde{\Omega}) \) with \( v \in I = [v_1, v_2] \), is calculated via

\[
G_T = \int_I dv \int_{S^2} d\tilde{\Omega} g_1(v)g_2(\tilde{\Omega}) F(v, \tilde{\Omega}). \tag{9}
\]

Since

\[
F_q(v, \tilde{\Omega}) = \sum_{i=I(L(q-1)+1)}^{L_q} \psi_i(v) \varphi_i(\tilde{\Omega}), \tag{10}
\]

is the solution to the Boltzmann equation in the macro-interval \( I_q^M \) based on the method of weighted residuals, we can write the sub-macroscopic quantity \( G_{qT} \) with \( v \in I_q^M = [w_{L(q-1)}, w_{Lq}] \) and consequently the total \( G \) with \( v \in [0, \infty) \) in the form

\[
G_{qT} = \sum_{i=I(L(q-1)+1)}^{L_q} \int_{w_{i-1}}^{w_i} dv g_1(v) \psi_i(v) \int_{S^2} d\tilde{\Omega} g_2(\tilde{\Omega}) \varphi_i(\tilde{\Omega}) \tag{11}
\]

Thus, the smaller we choose \( L \), the more sub-macroscopic quantities \( G_{qT} \) are obtained, which leads to more information about the macroscopic quantity \( G \). It should be noted that an individual \( \varphi_i \) does not have an immediate physical meaning, but only quantities related to \( \varphi_i \) via (10) and (11). To sum up, we state that \( L = 3 \) is certainly a proper choice for a realistic description of a fluid dynamic situation.
THE $P_2$ APPROXIMATION

In this section, we describe a method for treating the angular depending functions $\varphi_i(\hat{\Omega})$ in the revised multigroup approach. This $P_2$ approximation [9] consists of the expansion of these functions in spherical harmonics and the truncation of these series after the third term. Hence, we obtain an ansatz for the $N$ unknown functions

$$\varphi_i(x, \hat{\Omega}, t) = \frac{1}{4\pi} \left( n_i(x, t) + 3J_i(x, t) \cdot \hat{\Omega} + \frac{15}{2} \mathbb{C}_i(x, t) : (\hat{\Omega} \otimes \hat{\Omega}) \right).$$

This ansatz implies that

$$n_i = \int_{S^2} d\hat{\Omega} \varphi_i(\hat{\Omega}), \quad J_i = \int_{S^2} d\hat{\Omega} \hat{\Omega} \varphi_i(\hat{\Omega}), \quad \mathbb{C}_i = \int_{S^2} d\hat{\Omega} (\hat{\Omega} \cdot \hat{\Omega}) \varphi_i(\hat{\Omega}),$$

with

$$a \circ b = \frac{1}{2} (a \otimes b + b \otimes a) - \frac{1}{3} \mathbb{P} \cdot a.$$ 

We remark that $\mathbb{C}_i$ is a symmetric traceless tensor.

As shown in the previous section, the revised multigroup approach leads to $N$ evolution equations for the $N$ unknown functions $\varphi_i$ depending on the position $x$, the direction of velocity $\hat{\Omega} = v/\nu$ and the time $t$. The $P_2$ approximation yields an ansatz for these $\varphi_i$ with $9N$ new unknowns $n_i$, $J_i$ and $\mathbb{C}_i$ just depending on $x$ and $t$. The evolution equations for these $9N$ unknowns are obtained by projecting the multigroup system onto $1$, $\hat{\Omega}$ and $\hat{\Omega} \cdot \hat{\Omega}$. This means that one inserts the $P_2$ ansatz (13) into the multigroup approach (7), multiplies the result with $1$, $\hat{\Omega}$ and $\hat{\Omega} \cdot \hat{\Omega}$, respectively, and integrates it with respect to $\hat{\Omega}$ over the whole two-dimensional unit sphere $S^2$.

The revised multigroup approach linked with the $P_2$ approximation constitutes a system of evolution equations for the quantities $n_i$, $J_i$ and $\mathbb{C}_i$, which are moments of the distribution function. Hence, it is not surprising that the connection between these quantities and other speed moments, namely macroscopic quantities, is quite a simple one.

The comparison of the multigroup versions of particle density $n$, drift velocity $u$, momentum flux $K$, energy density $K$ and energy flux $Q$ (12) and (14) leads to the following expressions for macroscopic quantities in the RMA-$P_2$ formulation:

$$n = \sum_{i=1}^{N} n_i, \quad nu = \sum_{i=1}^{N} \langle v \rangle_i J_i, \quad \mathbb{K} = m \sum_{i=1}^{N} \langle v^2 \rangle_i \left( \mathbb{C}_i + \frac{n_i}{3} \mathbb{P} \right), \quad K = \frac{m}{2} \sum_{i=1}^{N} \langle v^2 \rangle_i n_i, \quad Q = \frac{m}{2} \sum_{i=1}^{N} \langle v^3 \rangle_i J_i.$$  

Finally, we try to transcript some macroscopic quantities into our multigroup scheme by splitting up the whole range of $v$ into a sum of intervals. Thus, we get for the particle density $n$, drift velocity $u$, momentum flux $K$ and energy flux $Q$:

$$n = \int_{\mathbb{R}^3} d^3 v f(v) = \sum_{i=1}^{N} \int_{S^2} d\hat{\Omega} \varphi_i(\hat{\Omega}),$$

$$nu = \int_{\mathbb{R}^3} d^3 v v f(v) = \sum_{i=1}^{N} \langle v \rangle_i \int_{S^2} d\hat{\Omega} \hat{\Omega} \varphi_i(\hat{\Omega}),$$

$$K = \frac{m}{2} \int_{\mathbb{R}^3} d^3 v v^2 f(v) = \frac{m}{2} \sum_{i=1}^{N} \langle v^2 \rangle_i \int_{S^2} d\hat{\Omega} \hat{\Omega} \varphi_i(\hat{\Omega}),$$

$$Q = \frac{m}{2} \int_{\mathbb{R}^3} d^3 v v^3 f(v) = \frac{m}{2} \sum_{i=1}^{N} \langle v^3 \rangle_i \int_{S^2} d\hat{\Omega} \hat{\Omega} \varphi_i(\hat{\Omega}).$$

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Moreover, we find for the stress tensor \( \mathbb{P} = \mathbb{K} - \rho \mathbf{u} \otimes \mathbf{u} \), the pressure \( p = 1/3 \text{tr} \mathbb{P} \) and the temperature \( T = p/(nk_B) \) separating parts of ordered motion and peculiar velocity

\[
\mathbb{P} = m \sum_{i=1}^{N} \left[ \langle \dot{v}^2 \rangle_i \left( C_i + \frac{n_i}{3} \right) - \langle v \rangle_i \sum_{j=1}^{N} \langle v \rangle_j \mathbf{J}_j / n \right], \tag{17}
\]

\[
p = \frac{m}{3} \sum_{i=1}^{N} \left( \langle \dot{v}^2 \rangle_i - \langle v \rangle_i \right) \sum_{j=1}^{N} \langle v \rangle_j \mathbf{J}_j / n, \tag{18}
\]

\[
T = \frac{m}{3k_B} \sum_{i=1}^{N} \left( \langle \dot{v}^2 \rangle_i - \langle v \rangle_i \right) \sum_{j=1}^{N} \langle v \rangle_j \mathbf{J}_j / n. \tag{19}
\]

As a consequence of the used multigroup concept, our model provides not only average values for macroscopic quantities to the whole velocity space but it also contains the information about the sub-macroscopic quantities to different speed macro-intervals as well (cp. (11)). Thus, the particle number \( \tilde{n}_q \) and the drift velocity \( \tilde{u}_q \) due to particles in the macro-interval \( v \in I_q^M = [w_{Lq-1}, w_{Lq}], q = 1, 2, \ldots N/L \) are obtained by

\[
\tilde{n}_q = \sum_{i=L(q-1)+1}^{Lq} n_i, \quad \tilde{u}_q = \frac{1}{n} \sum_{i=L(q-1)+1}^{Lq} \langle v \rangle_i \mathbf{J}_i. \tag{20}
\]

The above equations indicate a simple and elegant way to compare results of the RMA-P_{2} approximation with exact solutions \( f_A(v) \) to the Boltzmann equation. The RMA-P_{2} formulations of macroscopic quantities (16) have their counterparts in the first terms of (12). The particle densities \( \tilde{n}_q \) and drift velocities \( \tilde{u}_q \) referring to the speed macro-intervals for the given distribution function can be calculated by the integrations

\[
\tilde{n}_q = \int_{Q} dv v^2 \int_{S^2} d\hat{\Omega} f_A(v \hat{\Omega}), \quad \tilde{u}_q = \frac{1}{n} \int_{Q} dv v^3 \int_{S^2} d\hat{\Omega} \hat{v} f_A(v \hat{\Omega}). \tag{21}
\]

**CONSERVATION LAWS**

Now, we prove that the revised multigroup \( P_{3} \) approach preserves particle number, momentum and energy density for arbitrary choice of \( N \) and shape functions \( \psi_i(v) \), provided \( L \geq 3 \). First of all, we state that

\[
\int_{\mathbb{R}^3} d^3 v h(v \hat{\Omega}) \Pi(v' \hat{\Omega}', v'' \hat{\Omega}'' \rightarrow v \hat{\Omega}) = \frac{1}{2} [h(v' \hat{\Omega}') + h(v'' \hat{\Omega}'')], \tag{22}
\]

when \( h(v) \) is a collision invariant [7]. The multiplication with \( g(|v' \hat{\Omega}' - v'' \hat{\Omega}''|) f(v' \hat{\Omega}') f(v'' \hat{\Omega}'') \) and the integration of the result with respect to \( v' \) and \( v'' \) ends in

\[
\int_0^{\infty} dv' \int_0^{\infty} dv'' \int_{S^2} d\hat{\Omega}' \int_{S^2} d\hat{\Omega}'' \int_0^{\infty} dv \int_{S^2} d\hat{\Omega} h(v \hat{\Omega}) g(|v' \hat{\Omega}' - v'' \hat{\Omega}''|) \Pi(v' \hat{\Omega}', v'' \hat{\Omega}'' \rightarrow v \hat{\Omega}) F(v', \hat{\Omega}') F(v'', \hat{\Omega}'') \tag{23}
\]

\[
= \int_0^{\infty} dv' \int_0^{\infty} dv'' \int_{S^2} d\hat{\Omega}' \int_{S^2} d\hat{\Omega}'' \frac{1}{2} g(|v' \hat{\Omega}' - v'' \hat{\Omega}''|) [h(v' \hat{\Omega}') + h(v'' \hat{\Omega}'')] F(v', \hat{\Omega}') F(v'', \hat{\Omega}''). \tag{24}
\]

When the integrations with respect to \( v, v' \) and \( v'' \) are split up into sums of integrations over speed intervals \( I_i = [w_{i-1}, w_i] \), we obtain

\[
\sum_{i,j,k=1}^{N} \int_{I_i} dv' \int_{I_j} dv'' \int_{S^2} d\hat{\Omega}' \int_{S^2} d\hat{\Omega}'' \int_0^{\infty} dv \int_{S^2} d\hat{\Omega} h(v \hat{\Omega}) g(|v' \hat{\Omega}' - v'' \hat{\Omega}''|) \Pi(v' \hat{\Omega}', v'' \hat{\Omega}'' \rightarrow v \hat{\Omega}) F(v', \hat{\Omega}') F(v'', \hat{\Omega}'') \tag{25}
\]

\[
= \sum_{i,j=1}^{N} \int_{I_i} dv' \int_{I_j} dv'' \int_{S^2} d\hat{\Omega}' \int_{S^2} d\hat{\Omega}'' \frac{1}{2} g(|v' \hat{\Omega}' - v'' \hat{\Omega}''|) [h(v' \hat{\Omega}') + h(v'' \hat{\Omega}'')] F(v', \hat{\Omega}') F(v'', \hat{\Omega}''). \tag{26}
\]
Finally, the insertion of the multigroup ansatz (4) into the above equation and the application of the definitions (8) of \( g_{ij}(\hat{\Omega}, \hat{\Omega}^\prime) \) and \( \Pi_{jk}^{(l)}(\hat{\Omega}, \hat{\Omega}^\prime, \hat{\Omega}^\prime) \rightarrow \hat{\Omega} \) yields for \( h_1(\nu \hat{\Omega}) = 1 \)

\[
\sum_{i,j,k=1}^{N} \int_{\mathcal{S}^2} d\hat{\Omega} \int_{\mathcal{S}^2} d\hat{\Omega}^\prime \int_{\mathcal{S}^2} d\hat{\Omega}^\prime \left[ g_{ij}^{(0)}(\hat{\Omega}, \hat{\Omega}^\prime) \Pi_{jk}^{(0)}(\hat{\Omega}, \hat{\Omega}^\prime) \varphi_j(\hat{\Omega}) \varphi_k(\hat{\Omega}^\prime) \right] = \sum_{i,j=1}^{N} \int_{\mathcal{S}^2} d\hat{\Omega} \int_{\mathcal{S}^2} d\hat{\Omega}^\prime \left[ g_{ij}^{(0)}(\hat{\Omega}, \hat{\Omega}^\prime) \varphi_i(\hat{\Omega}) \varphi_j(\hat{\Omega}^\prime) \right].
\]

Moreover, we obtain for \( h_2(\nu \hat{\Omega}) = \nu^2 \frac{\nu}{2} \) end in

\[
\sum_{i,j,k=1}^{N} \int_{\mathcal{S}^2} d\hat{\Omega} \int_{\mathcal{S}^2} d\hat{\Omega}^\prime \int_{\mathcal{S}^2} d\hat{\Omega}^\prime \left[ g_{ij}^{(1)}(\hat{\Omega}, \hat{\Omega}^\prime) \Pi_{jk}^{(1)}(\hat{\Omega}, \hat{\Omega}^\prime) \varphi_j(\hat{\Omega}) \varphi_k(\hat{\Omega}^\prime) \right] = \sum_{i,j=1}^{N} \int_{\mathcal{S}^2} d\hat{\Omega} \int_{\mathcal{S}^2} d\hat{\Omega}^\prime \left[ g_{ij}^{(1)}(\hat{\Omega}, \hat{\Omega}^\prime) \varphi_i(\hat{\Omega}) \varphi_j(\hat{\Omega}^\prime) \right].
\]

while the calculations for \( h_3(\nu \hat{\Omega}) = \nu^2/2 \) end in

\[
\sum_{i,j,k=1}^{N} \int_{\mathcal{S}^2} d\hat{\Omega} \int_{\mathcal{S}^2} d\hat{\Omega}^\prime \int_{\mathcal{S}^2} d\hat{\Omega}^\prime \left[ g_{ij}^{(2)}(\hat{\Omega}, \hat{\Omega}^\prime) \Pi_{jk}^{(2)}(\hat{\Omega}, \hat{\Omega}^\prime) \varphi_j(\hat{\Omega}) \varphi_k(\hat{\Omega}^\prime) \right] = \sum_{i,j=1}^{N} \int_{\mathcal{S}^2} d\hat{\Omega} \int_{\mathcal{S}^2} d\hat{\Omega}^\prime \left[ g_{ij}^{(2)}(\hat{\Omega}, \hat{\Omega}^\prime) \varphi_i(\hat{\Omega}) \varphi_j(\hat{\Omega}^\prime) \right].
\]

Finally, we manipulate the revised multigroup approach in the following way. For \( l = 0 \) we project the equations with respect to \( \hat{\Omega} \) onto 1 and sum the result over all the possible \( q \). Hence, we obtain by the help of (14) and (16)

\[
\frac{\partial}{\partial t} \left( \sum_{i=1}^{N} \int_{\mathcal{S}^2} d\hat{\Omega} \varphi_i(\hat{\Omega}) \right) + \nabla \cdot \left( \sum_{i=1}^{N} \langle v^2 \rangle_i \int_{\mathcal{S}^2} d\hat{\Omega} \varphi_i(\hat{\Omega}) \right) = \frac{\partial}{\partial t} \left( \sum_{i=1}^{N} n_i \right) + \nabla \cdot \left( \sum_{i=1}^{N} \langle v_i \rangle_i J_i \right) = 0,
\]

and therefore

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n u) = 0,
\]

since the right-hand side of this equation vanishes because of (23). If one takes \( l = 1 \) in (7), projects this equations onto \( m \hat{\Omega} \), sums the result over the index \( q \) and applies (24), the validity of the following equation is proved with (14) and (16):

\[
\frac{\partial}{\partial t} \left( m \sum_{i=1}^{N} \langle v_i \rangle_i \int_{\mathcal{S}^2} d\hat{\Omega} \varphi_i(\hat{\Omega}) \right) + \nabla \cdot \left( m \sum_{i=1}^{N} \langle v^2 \rangle_i \int_{\mathcal{S}^2} d\hat{\Omega} \varphi_i(\hat{\Omega}) \right) = \frac{\partial}{\partial t} \left( m \sum_{i=1}^{N} n_i \right) + \nabla \cdot \left( m \sum_{i=1}^{N} \langle v^2 \rangle_i \left( C_i + \frac{n_i}{2} I_3 \right) \right) = 0,
\]

which means that

\[
\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u) = 0.
\]

Finally, we use \( l = 2 \) for constructing a further equation by the projection onto \( m/2 \). Taking advantage of (25), (14) and (16) yields

\[
\frac{\partial}{\partial t} \left( \frac{m}{2} \sum_{i=1}^{N} \langle v^2 \rangle_i \int_{\mathcal{S}^2} d\hat{\Omega} \varphi_i(\hat{\Omega}) \right) + \nabla \cdot \left( \frac{m}{2} \sum_{i=1}^{N} \langle v^2 \rangle_i \int_{\mathcal{S}^2} d\hat{\Omega} \varphi_i(\hat{\Omega}) \right) = \frac{\partial}{\partial t} \left( \frac{m}{2} \sum_{i=1}^{N} \langle v^2 \rangle_i n_i \right) + \nabla \cdot \left( \frac{m}{2} \sum_{i=1}^{N} \langle v^2 \rangle_i J_i \right) = 0
\]
FIGURE 2. The initial (left plot) and equilibrium distribution (right plot) of the particle density \( \hat{n}_q \) in comparison to \( \tilde{n}_q \) obtained from the Maxwell distribution.

FIGURE 3. The initial (left plot) and equilibrium distribution (right plot) of the drift velocity \( \hat{u}_{q,i} \) in comparison to \( \tilde{u}_{q,i} \) obtained from the Maxwell distribution.

and can be written as

\[
\frac{\partial K}{\partial t} + \nabla \cdot \mathbf{Q} = 0. \tag{31}
\]

The equations (26), (28) and (30) are the multigroup versions of conservation laws for mass, momentum and energy. Since they agree completely with the conservation equations of the original Boltzmann equation, our model can be regarded as quite realistic in the sense that mass, momentum and energy conservation are described in a correct way.

**NUMERICAL RESULTS**

In this section, we demonstrate the relaxation of homogeneous gases from an arbitrary initial distribution towards the equilibrium state by the help of the deduced RMA-\( P_2 \) model.

Here, we define the speed-knots simply by demanding an equidistant partition of the speed axis:

\[
w_j = v_1 \left( i + \frac{1}{2} \right), \quad i = 1, 2, \ldots, N - 1. \tag{32}
\]

Since we want to neglect the influence of particles with speeds higher than \( w_N \), we set \( w_N \) to a sufficient large number. Moreover, we choose delta-distributions as shape functions:

\[
\psi_i(v) = \delta(v - v_i), \\
v_i = v_1 i, \quad i = 1, 2, \ldots, N. \tag{33}
\]

Because of the definition of the Dirac delta distribution, one can easily prove that these \( \psi_i(v) \) match with the conditions (5) for shape functions. We assume that our gas particles are Maxwell molecules, which lead to constant collision frequency.

We intend to illustrate the physical behavior of our model by comparing the equilibrium state with a multigroup representation of the Maxwell distribution. For this purpose, we study the relaxation of a drifting gas with non-vanishing initial shear stress. Thereby, we choose \( N = 21 \) speed intervals, which are combined to \( N/L = 7 \) macro-intervals.
The left plots in Figs. 2, 3 and 4 illustrate the initial distribution of the particle density $\hat{n}_q$ and the drift velocity $\hat{u}_q$ (18) referring to the macro-intervals $I_q^M = [w_{L(q-1)}, w_{Lq}]$ and the initial stress tensor $\mathbb{P}$ (17). The comparison of these initial conditions with the Maxwellian allows us to regard the present initial state as a far from equilibrium state.

The evolution equations for the quantities $n_i$, $J_i$ and $K_i$ are numerically solved. After a sufficiently long time, the variations of these quantities vanish and an equilibrium state is reached.

The right plots in Figs. 2 and 3 show the particle densities $\hat{n}_q$ and the drift velocities $\hat{u}_q$ in the equilibrium state, compared with $\tilde{n}_q$ and $\tilde{u}_q$ from the Maxwell distribution via (19). The necessary parameters for the Maxwellian are obtained from the initial conditions of $n_i$, $J_i$ and $C_i$ by (16) and (17). The figures illustrate the agreement of the equilibrium solution of the RMA-P$_2$ model with the Maxwellian. The particle densities and drift velocities match within the range of numerical accuracy with the results obtained by the integration of the Maxwellian. The right plot in Fig. 4 shows the final pressure tensor. The initial shear stress components vanish, while the diagonal components of the stress tensor even and are equal to the pressure in the regarded gas.

The most important properties of our RMA-P$_2$ model are the exact conservation of particle density, drift velocity and energy density combined with great flexibility in the partition of the speed axis and the choice of the shape functions. Thus, the RMA-P$_2$ model is a very suitable tool to describe the properties of rarefied gases.

REFERENCES