Exact Solutions for the n-axis and Spin Tune in Model Storage Rings

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Abstract. We present a new nonperturbative formalism MILES to calculate the n-axis in storage rings. We employ MILES to obtain the exact solution for the n-axis in several model storage rings. In particular, we display the exact analytical solution for the single resonance model with one Snake. Our solution depends on new types of mathematical function, which we call "sine-factorial" and "sine-Bessel" functions. We confirm the spectrum of Snake resonances found by Lee and Tepikian. Also, under suitable circumstances, we show that the spin tune depends explicitly on the orbital phase. We term such trajectories "exceptional orbits."

INTRODUCTION

We present a new, nonperturbative formalism "MILES" to calculate the \( \hat{n} \) axis in storage rings. The name is simply an anagram of the author's old algorithm and program SMILE [1]. We employ MILES to obtain the exact analytical solution for the \( \hat{n} \) axis for several storage ring models. The models contain Siberian Snakes. In particular, we solve the model of a planar ring with a single resonance driving term (the "single resonance model") and one Snake (also two, four, etc. Snakes). This model is in some sense the "classic" problem in the field. We also solve the same model using Yokoya's formalism SODOM2 [2]. Our solution contains new types of mathematical functions, which we call "sine-factorial" and "sine-Bessel" functions. We confirm a longstanding conjecture by Yokoya [3] that the spin tune is 1/2 even off-axis for a single resonance model with multiple Snakes. We verify the spectrum of the Snake resonances found by Lee and Tepikian [4]. We show they are higher order depolarizing resonances, following from the standard spin resonance condition. Finally, we show that under suitable circumstances the spin tune can depend explicitly on the orbital phase. We term such a trajectories "exceptional orbits."

MILES ALGORITHM

For brevity, denote a point in the orbital phase space by \( z \). By definition, the \( \hat{n} \) axis transforms as a vector field over the orbital phase space, i.e.

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\[
\vec{\sigma} \cdot \vec{n}(z_f) = M \vec{\sigma} \cdot \vec{n}(z_i) M^{-1}.
\]

Here $M$ is the spin-orbit map from the initial to the final azimuth. It is simplest to choose $M$ to be the one-turn map. Then we can write, at a fixed azimuth $\theta_*$,

\[
\vec{\sigma} \cdot \vec{n}(\vec{\phi}_* + \vec{\mu}) = M \vec{\sigma} \cdot \vec{n}(\vec{\phi}_*) M^{-1}.
\]

We denote the orbital action-angle variables by $(\vec{I}, \vec{\phi})$ and the orbital tunes by $Q_j$ $(j=1,2,3)$. The one-turn orbital phase advances are $\mu_j = 2\pi Q_j$. We parameterize the map $M$ via

\[
M = \begin{pmatrix}
  f & -g^* \\
  g & f^*
\end{pmatrix}.
\]

We employ a basis of right-handed orthonormal vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, which are radial, longitudinal and vertical, respectively. The components of $\vec{n}$ in this basis are $(n_1, n_2, n_3)$ and we define $n_\pm = n_1 \pm i n_2$ (so $n_\mp = n_1^* \mp i n_2^*$). This yields the equations

\[
\begin{align*}
  n_3(\vec{\phi}_* + \vec{\mu}) &= (f^* - gg^*)n_3(\vec{\phi}_*) - f^* g^* n_2(\vec{\phi}_*) - f g n_1(\vec{\phi}_*) \\
  n_1(\vec{\phi}_* + \vec{\mu}) &= 2f^* g n_3(\vec{\phi}_*) + f^* n_2(\vec{\phi}_*) - g^* n_3(\vec{\phi}_*) - g n_2(\vec{\phi}_*)
\end{align*}
\]

We now expand in Fourier harmonics

\[
\begin{align*}
  n_3(\vec{\phi}_*) &= \sum_m n_{3m} e^{im\vec{\phi}_*} \\
  n_1(\vec{\phi}_*) &= \sum_m n_{1m} e^{im\vec{\phi}_*}
\end{align*}
\]

We must also expand $f$ and $g$ in Fourier harmonics. We then solve for the Fourier coefficients $n_{3m}$ and $n_{1m}$. This is the MILES algorithm. We obtain the solution at other azimuths by tracking.

We have employed MILES to obtain analytical solutions for the following models.

- Single resonance model (SRM)
- SRM with partial Type 3 Snake
- Planar ring with vertical resonance driving term and one or two (or more) Snakes
- SRM with one Snake
- SRM with two or more Snakes

We shall present the explicit solution for the SRM with one Snake below.

**SRM WITH ONE SNAKE**

We consider a model of a planar ring with a single resonance driving term, of strength $e$, in the horizontal plane and a single Snake. The Snake is located at $\theta_s = 0$ and its axis points at an angle $\xi$ relative to $\hat{e}_1$. The spin precession vector is

\[
\vec{W} = \nu_s \hat{e}_1 + \epsilon (\cos \phi \hat{e}_1 + \sin \phi \hat{e}_2) + \pi \delta_\nu (\theta) (\cos \xi \hat{e}_1 + \sin \xi \hat{e}_2).
\]

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Here \( \nu_0 \) is a constant and \( \delta_p(\theta) \) is the periodic delta function:

\[
\delta_p(\theta) = \sum_{j=-\infty}^{\infty} \delta(\theta - 2\pi j).
\]

We define \( \Omega = \sqrt{(\nu_0 - Q)^2 + e^2} \) and \( \eta = (e/\Omega)\sin(\pi\Omega) \), so \(-1 \leq \eta \leq 1\). We place the origin just after the Snake. Then the one-turn map is

\[
M = \begin{pmatrix}
-\eta e^{-i(\Lambda - \xi + \mu/2)} & -i\sqrt{1-\eta^2} e^{i(\Lambda - \xi + \mu/2)} \\
-i\sqrt{1-\eta^2} e^{-i(\Lambda + \xi + \mu/2)} & -\eta e^{i(\Lambda + \xi + \mu/2)}
\end{pmatrix}
\]

where

\[
\cos(\pi\Omega) \pm \frac{\nu_0 - Q}{\Omega} \sin(\pi\Omega) = \sqrt{1-\eta^2} e^{\pm i\kappa}
\]

We also set

\[
\begin{align*}
    n_h &= (1 - \eta^2) a \\
    n_e &= i g b
\end{align*}
\]

We expand in Fourier harmonics

\[
\begin{align*}
a(\phi_\star) &= 2 \sum_{m\text{ odd}} a_m \sin(m(\phi_\star - \xi)) \\
b(\phi_\star) &= b_0 + 2 \sum_{m\text{ even}} b_m \cos(m(\phi_\star - \xi))
\end{align*}
\]

Notice that the expression for \( a \) contains only odd harmonics, all sines, and \( b \) contains only even harmonics, all cosines. This is a nice pattern. To solve for the Fourier coefficients, we define \( \delta = Q - 1/2 \) and introduce "sine-factorial" functions

\[
\begin{align*}
    S_m(\delta) &= \sin(\pi \delta) \sin(2\pi \delta) \cdots \sin(m\pi \delta) \\
    C_m(\delta) &= \cos(\pi \delta) \cos(2\pi \delta) \cdots \cos(m\pi \delta)
\end{align*}
\]

We also adopt the convention \( S_0(\delta) = C_0(\delta) = 1 \). Then the solution is

\[
\begin{align*}
a_m &= \frac{1}{\cos(m\pi\delta/2)} \sum_{k=0}^{C_{m/2+k}^2(2\delta)} S_k(2\delta)S_{m-k}(2\delta)(-1)^k \eta^{m+2k} \\
b_m &= \sum_{k=0}^{C_{m-1/2+k}^2(2\delta)} S_k(2\delta)S_{m+k}(2\delta)(-1)^k (\eta e^{i\delta})^{m+2k}
\end{align*}
\]

These solutions bear a close resemblance to the power series expansion of a Bessel function

\[
J_m = \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!}(-1)^k \left( \frac{\eta}{2} \right)^{m+2k}
\]

We also display the solution using the SODOM2 [2] algorithm. This algorithm actually solves for a spinor. We write \( \hat{n} = \Psi^* \tilde{\sigma} \Psi \) and parameterize

\[
\Psi = \begin{pmatrix} A \\ i g B \end{pmatrix}
\]
We set

\[ A(\phi) = A_0 + 2 \sum_{m=\text{even}} A_m \cos(m(\phi - \xi)) + 2 \sum_{m=\text{odd}} A_m \sin(m(\phi - \xi)) \]

\[ B(\phi) = B_0 + 2 \sum_{m=\text{even}} B_m \cos(m(\phi - \xi)) - 2 \sum_{m=\text{odd}} B_m \sin(m(\phi - \xi)) \]

(16)

The minus sign in the expression for \( B \) is significant. It is not a misprint. The answer is

\[ A_m = \sum_{k=0}^{\infty} \frac{e^{i k (m+k) \eta s}}{S_k(\delta) S_{m+k}(\delta)} (-1)^k \left( \frac{\eta e^{-i s}}{2} \right)^{m+2k} \]

\[ B_m = \sum_{k=0}^{\infty} \frac{e^{i k (m+k) \eta s}}{S_k(\delta) S_{m+k}(\delta)} (-1)^k \left( \frac{\eta e^{i s}}{2} \right)^{m+2k} \]

(17)

These solutions have a nicer symmetry than the solutions for the vector \( \mathbf{\hat{n}} \). We call the above functions "sine-Bessel" functions. However, the elegant pattern of the Fourier harmonics in \( \mathbf{\hat{n}} \) is not so obvious from the spinor solution. Along with sine-factorials, the sine-Bessel functions are new mathematical functions. They do not appear to be in the mathematical literature.

The SODOM2 formalism also calculates the spin tune. The solution is \( \nu = 1/2 \), not only on the closed orbit, but also off-axis, for all nonresonant orbits with \( |\eta| < 1 \). We also find that \( \nu = 1/2 \) off-axis for a model with 2 or more Snakes (provided \( |\eta| < 1 \)). This confirms Yokoya's [3] conjecture, that \( \nu = 1/2 \) off-axis for nonresonant orbits in a planar ring with multiple Snakes and a single resonance driving term.

**SNAKE RESONANCES**

Depolarizing spin resonances occur whenever there are zero denominators in the solution for \( \mathbf{\hat{n}} \). For a ring with two Snakes, such zeroes occur whenever the orbital tune has the form

\[ Q = \frac{2k - 1}{2(2m - 1)}. \]

(18)

Here \( m \) and \( k \) are integers. This is precisely the spectrum of the so-called "Snake resonances" found by Lee and Tepikian [4]. We therefore verify that they obtained the correct resonance spectrum. We also see that Snake resonances are higher-order spin resonances and follow from the standard formula

\[ \nu + m'Q = k. \]

(19)

We substitute \( \nu = 1/2 \) (which we have shown is the value off-axis) to obtain

\[ Q = \frac{2k - 1}{2m'}. \]

(20)

A more detailed analysis of the Fourier structure of the one-turn map is required to show that \( m' \) must be an odd integer \( m' = 2m - 1 \), which completes the proof.
EXCEPTIONAL ORBITS

The remaining case to consider is $|\eta|=1$. For $\eta = -1$, the one-turn map is

$$M = \begin{pmatrix} e^{-i(\phi_0 - \xi + \mu/2)} & 0 \\ 0 & e^{i(\phi_0 - \xi + \mu/2)} \end{pmatrix}. \tag{21}$$

The solution is clearly $\hat{n} = \hat{e}_3$. However the spin tune is not 1/2 but is

$$\nu = \frac{\phi_0 - \xi + \mu/2}{\pi}. \tag{22}$$

The spin tune depends explicitly on the orbital phase. We call such a trajectory an "exceptional orbit." On an exceptional orbit, the Stern-Gerlach force cannot be neglected. It leads to a secular growth in the amplitudes of the orbital motion. This invalidates the semiclassical approximation normally employed in accelerator physics.

CONCLUSION

We have presented a new nonperturbative formalism "MILES" to calculate the $\hat{n}$ axis in storage rings. We employed it to solve several models analytically. We displayed the analytical solution for a planar ring with a single resonance driving term and one Snake. The solution contained some new mathematical functions, which we call "sine-factorial" and "sine-Bessel" functions. These functions do not appear to be in the mathematical literature. We showed that the spin tune is 1/2 even off-axis, and we also derived the spectrum of the Snake resonances. Finally, we showed that under some circumstances the spin tune depends explicitly on the orbital phase. We termed such trajectories "exceptional orbits." The semiclassical approximation of accelerator dynamics breaks down on such orbits.

REFERENCES