A fluid description of kinetic effects for Alfvén wave trains

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Abstract. A generalized Kinetic Derivative Nonlinear Schrödinger equation for the multidimensional dynamics of Alfvén wave trains is derived from the Vlasov-Maxwell equations. It includes the coupling to the mean effect of the longitudinal fields averaged along the direction of propagation, that plays a significant role in the transverse dynamics of extended wave packets, in particular in the phenomenon of transverse collapse and Alfvén wave filamentation. This equation can be viewed as an asymptotically exact fluid model including the Landau damping, that can be used as a benchmark for more general magnetohydrodynamic type descriptions of collisionless plasmas.

INTRODUCTION

Astrophysical and geophysical plasmas are usually permeated by an ambient magnetic field, leading to a strong anisotropy of the observed fluctuations. In the case of incompressible fluids, the dynamics is, to leading order in an expansion in terms of the magnitude of the external field, two-dimensional in planes perpendicular to the ambient field with a coupling between these planes that reduces to linear Alfvén waves [1, 2]. The longitudinal transfer is subdominant and the energy spectrum displays a rapid decay in this direction. Furthermore, the nature of the transverse turbulence is prescribed by the ratio of the local Alfvén and eddy-turnover times. In the range of scales for which the Alfvén time is the shortest, a weak turbulence regime is expected, characterized by a \( k^{-2} \) energy spectrum [3, 4]. In contrast, in the spectral domain where the opposite ordering holds, the turbulence is strong with a spectrum whose behavior is still debated between \( k^{-5/3} \) and \( k^{-3/2} \) [5, 6, 7, 8]. In the \((k_x, k_y)\)-plane, the two regimes are separated by a “critical curve” of the form \( k_y = k_x^{2/3} \), according to the Goldreich-Shridhar theory [9].

The presence of compressibility can alter this picture. As long as the Alfvénic Mach number is small, the system is governed by the reduced MHD equations, where the transverse dynamics is to leading order incompressible [10, 11]. This dynamics is purely 2D when the \( \beta \) of the plasma, defined as the square ratio of the sound and Alfvén speeds, is far from unity, while it becomes 2.5D as \( \beta \approx 1 \) is approached. The reduced MHD regimes do not necessarily require that the longitudinal characteristic scale be much larger than the transverse one. Indeed, as shown in [12], the derivatives of the longitudinal fields along the ambient magnetic field are comparable to the transverse gradients of the perpendicular components. This situation results from the presence of nonlinear Alfvén waves that, for parallel propagation in a specific direction, are governed by the Derivative Nonlinear Schrödinger equation (DNLS) and can coexist with the transverse turbulence without affecting it, at least when \( \beta \) is far from unity. When Alfvén waves are propagating in both directions, they contribute to feed the transverse dynamics by the usual coupling between contra-propagating Alfvén pulses. In this case, even if not initially present, backward propagating waves will be created by the decay instability when \( \beta < 1 \) [13, 14, 15].

The Alfvén wave dynamics results from the competition between nonlinear, dispersive and dissipative processes. The first two effects were extensively studied in the past few years. As an example, a possibly important process that can affect the weak turbulence dynamics results from the filamentation instability [16] that leads to transverse collapse of a weakly nonlinear quasi-monochromatic dispersive Alfvén wave train and formation of intense magnetic filaments [17]. This transverse dynamics is essentially governed by the two-dimensional nonlinear Schrödinger equation for the amplitude of the wave. This envelope equation admits solutions that blow up in a finite time [18, 19]. As shown in [20], this asymptotic description provides a quantitatively accurate description of the early stages of the collapse. The influence of the oblique instabilities is also captured in this formalism, by retaining the coupling to magnetosonic waves. When such instabilities can develop, they tend to weaken the process of wave energy concentration, while enhancing the development of density gradients. Note
that a transverse concentration of wave energy is still observed in the numerical simulations, for Alfvén waves of moderate amplitude (that are not amenable to an envelope description), leading to the formation of helicoidal magnetic filaments [21]. The possibility for such structures to form in a context of wave turbulence and their effect on the spectral transfer of energy is an important question that has not yet been addressed.

Concerning dissipation in a collisionless medium, the proper mechanisms to be considered originate from the Landau damping of the ion acoustic waves and from the ion cyclotron resonance (when considering small enough scales). A quantitative description of such effects is delicate since a fluid formalism neglecting kinetic effects is not appropriate, while the use of the full Vlasov-Maxwell equations is not too large compared to the ion gyro-radius, especially for \( \beta \) of order one or larger [22]. This approach nevertheless fails when considering much longer waves or smaller amplitudes. The question thus arises of developing asymptotic approaches that incorporate the kinetic effects that are the most important for the dynamics.

**THE KINETIC DNLS EQUATION**

The dynamics of Alfvén waves propagating along a strong ambient field are amenable to an asymptotic expansion, directly from the Vlasov-Maxwell equation, when involving scales that are large compared to the ion Larmor radius and amplitudes small enough to keep linear dispersive effects relevant [23]. For this purpose, one writes the Vlasov-Maxwell equations in the form

\[
\partial_t F_r + \mathbf{v} \cdot \nabla F_r + \frac{q_r}{m_r} (E + \mathbf{v} \times \mathbf{B}) \nabla_x F_r = 0
\]

(1)

\[
\frac{1}{c} \partial_t B = -\nabla \times E
\]

(2)

\[
\nabla \times B = 4\pi \sum_q \varepsilon_q n_r \int v F_r d^3v
\]

(3)

\[
4\pi \sum_q \varepsilon_q n_r \int F_r d^3v = 0
\]

(4)

where the subscript \( r \) refers to the particle species and where the displacement current has been neglected.

For an ambient field of strength \( B_0 \) pointing in the \( x \)-direction, the approach consists in rescaling the longitudinal variable \( \xi = \mathbf{v}^2 (x - \lambda t) \), the transverse ones as \( \eta = \mathbf{v}^2 y \) and \( \zeta = \mathbf{v}^2 z \), and in introducing a time scale \( \tau = \mathbf{v}^4 t \). One also expands the distribution function and the electric and magnetic fields in the form

\[
F_r = F_r^{(0)} + \varepsilon (f_r^{(1)} + e f_r^{(1)} + \ldots)
\]

(5)

\[
B = B_0 + \varepsilon (b^{(0)} + e b^{(1)} + \ldots)
\]

(6)

\[
E = (e^{(0)} + \mathbf{v}^{(1)} + \ldots).
\]

(7)

Inserting this long-wave expansion into the Vlasov-Maxwell equations and averaging over the velocity, one selects the Alfvén waves, characterized by the correlation \( b_0^{(0)} = \frac{B_0}{2} u_0 \) between the transverse components of the magnetic field and of the hydrodynamic velocity \( u_0 = \frac{1}{\rho_0} \sum m_r n_r \int \mathbf{v}_r f_r^{(0)} d^3v \), where \( \rho^{(0)} = \sum m_r n_r \int F_r^{(0)} d^3v \).

At this order, the longitudinal magnetic field is \( b_0^{(0)} = 0 \). The propagation velocity \( \lambda \) is given by

\[
\lambda^2 \rho_0^{(0)} = \frac{1}{4\pi} |B_0|^2 + p_0^{(0)} - p_1^{(0)},
\]

(8)

where the usual expression for the Alfvén velocity is affected by the anisotropy of the equilibrium pressure tensor, characterized by its longitudinal and transverse components

\[
p_0^{(0)} = \sum m_r n_r \int v_1^2 F_r^{(0)} d^3v
\]

(9)

\[
p_1^{(0)} = \sum m_r n_r \int \frac{v_1^2}{2} F_r^{(0)} d^3v
\]

(10)

The time evolution of the transverse magnetic field is obtained at a higher order of the expansion in the form

\[
\partial_t b_\perp^{(0)} + \frac{\delta}{2\Omega_\perp} \partial_{\xi \perp} (\hat{\xi} \times b_\perp^{(0)}) - \frac{B_0}{2\lambda \rho_0^{(0)}} \nabla_\perp \tilde{P} + \frac{1}{2\lambda \rho_0^{(0)}} \partial_{\xi \perp} (\tilde{P} b_\perp^{(0)}) + \partial_{\xi \perp} (\nabla \cdot b_\perp^{(0)}) = 0
\]

(11)

\[
\partial_{\xi \perp} b_\perp^{(1)} + \nabla_\perp b_\perp^{(0)} = 0
\]

(12)

with a dispersion coefficient \( \delta = \frac{1}{\rho_0^{(0)}} (\frac{b_0^{(0)}}{4\pi} + 2p_0^{(0)} - p_1^{(0)}) \) and fluctuations of total pressure in the direction transverse to the local magnetic field

\[
\tilde{P} = \frac{B_0^2}{4\pi} \tilde{A} + \tilde{P}_\perp^{(1)}.
\]

(13)

The magnetic contribution involves the magnetic field strength perturbation \( A = \frac{1}{\rho_0^{(0)}} |b_\perp^{(0)}|^2 + \frac{1}{\rho_0^{(0)}} b_\perp^{(1)} \) defined by expressing the amplitude of the total magnetic field in the form \( |B| = B_0 (1 + e\hat{A} + O(\varepsilon^2)) \). Separating the mean value in the longitudinal direction (denoted by \( \langle \cdot \rangle_\parallel \) from the fluctuations, one writes \( A = \langle |A| \rangle_\parallel + \hat{A} \). The fluctuations of transverse thermodynamical pressure

\[
\tilde{P}_\perp^{(1)} = (2p_0^{(0)} + N - M^2 L^{-1})\hat{A}
\]

(14)
are sensitive to the distortion of the local magnetic field produced by the wave and also includes the Landau damping effect through the operators

\[ L = 2\pi \Sigma \frac{\partial F[0]}{m_e n_r} \int_0^\infty \mathcal{G}_t d\left( \frac{v_t^2}{2} \right) \]  
\[ M = 2\pi \Sigma \xi n_r \int_0^\infty \frac{v_t^2}{2} \mathcal{G}_t d\left( \frac{v_t^2}{2} \right) \]  
\[ N = 2\pi \Sigma m_r n_r \int_0^\infty \frac{v_t^2}{4} \mathcal{G}_t d\left( \frac{v_t^2}{2} \right) \]

where, denoting by \( \mathcal{H} \) the Hilbert transform with respect to the \( \xi \) variable,

\[ \mathcal{G}_t = p.v. \int \frac{1}{v_t - \lambda} \frac{\partial F[0]}{\partial v_t} dv_t + \pi \frac{\partial F[0]}{\partial v_t} |_{v_t = \lambda} \mathcal{H} \]

Note that this operator has a symbol of order zero. As a result, all the scales are affected by the Landau damping. In particular, the criteria for modulational instability in the direction of propagation are strongly modified by this effect [24, 25] and new dissipative structures were also reported [26]. Note that the action of the Landau damping on the Alfvén waves is mediated by the coupling with ion-acoustic waves that are directly affected.

The above system is closed in the case of localized pulses that vanish as \( \xi \to \infty \) or, in the physical variables, at distances large compared with \( e^{-2} \), since in this case, \( \mathcal{H} = 0 \). These equations were derived by Register [23] using a Fourier space formalism. Mjølhus and Wyller [27] showed that the same equations can be obtained in a fluid formalism starting from an extension of the Hall-MHD equations including the electron pressure and finite Larmor radius effects [28]. They included the kinetic effects by estimating the transverse pressure fluctuation within the guiding center approximation.

### WAVE-TRAIN DYNAMICS AND FILAMENTATION

As noted in [29] and recently illustrated numerically on direct numerical simulations of the Hall-MHD equations [30], the filamentation instability, associated with the transverse collapse of the Alfvén wave, can only occur in the case of wave packets whose extension significantly exceeds the wave length of the carrier. In such a regime, the contributions of fields that only depend on the transverse coordinate are to be retained. They correspond to the \( \mathcal{H} \) term in eq. (11) that is computed by pushing the expansion to the next order, in the form

\[ \mathcal{H} = \langle u_1 \rangle \xi + \frac{\lambda}{2B_0} \langle b^{(1)} \rangle \xi + \frac{1}{\lambda} \rho^{(0)}(p^{(1)} - p^{(0)}) \langle A \rangle \xi \]

\[ + \frac{1}{2\lambda} \rho^{(0)}(p^{(0)} + p^{(1)}) \langle b^{(0)} \rangle \xi \]

The parallel hydrodynamic velocity averaged over the longitudinal coordinate, given to leading order by

\[ \langle u_1 \rangle \xi = \frac{1}{\rho^{(0)}} \sum_{m_r n_r} \int v_1 \langle f^{(1)} \rangle \xi d^3 v \]

obey

\[ \partial_t \langle u_1 \rangle \xi = \frac{1}{\rho^{(0)}B_0} \nabla \cdot \langle \tilde{p} b^{(0)} \rangle \xi \]

while the mean longitudinal magnetic field \( \langle b^{(1)} \rangle \xi \) is related to the mean density fluctuations (also constructed from \( \langle f^{(1)} \rangle \xi \)), by

\[ \frac{1}{\rho^{(0)}} \langle \tilde{p} b^{(0)} \rangle \xi = \frac{1}{B_0} \langle b^{(1)} \rangle \xi \]

One clearly sees that the mean fields are driven by the transverse gradients. Furthermore,

\[ \frac{B_0^2}{4\pi} \langle A \rangle \xi + \langle p^{(1)} \rangle \xi = \text{const.} \]

The system is completed by writing dynamical equations for the mean value of the pressure perturbations parallel and transverse to the local magnetic field, that to leading order read

\[ \langle p^{(1)} \rangle \xi = \sum_{m_r n_r} \int |v_1|^2 \langle f^{(1)} \rangle \xi d^3 v = \frac{B_0^2}{4\pi} \langle \tilde{p} b^{(0)} \rangle \xi - \lambda \rho^{(0)} \langle b^{(0)} \rangle \xi \]

The magnetic contributions in the above expressions reflects the distortion of the magnetic field lines. Dynamical equations governing the time evolution of these quantities are also obtained. They however seem too complicated to be explicitly written here and will be published elsewhere. The important point is that these equations complete the system that is then closed. It provides an asymptotically exact description of the nonlinear dynamics of Alfvén wave trains that retain the effect of the Landau damping mediated by the interaction with the ion-acoustic waves.

It is of interest to note the similarity (up to the Landau damping) between the above system and the equations derived from the Hall MHD equations for a polytropic gas where the pressure is assumed to be isotropic and to adiabatically follow the density variations. These equations read

\[ \tilde{P} = \frac{1}{1 - \beta} \frac{B_0^2}{4\pi} \lambda \]

(26)
\[
\frac{1}{\rho^{(0)}} \langle \rho^{(1)} \rangle_\xi = \frac{1}{B_0} \langle \rho^{(1)} \rangle_\xi
\]  
(27)

\[
\langle A \rangle_\xi + \frac{\beta}{1 + \beta} \frac{\langle \rho^{(1)} \rangle_\xi}{2B_0} = \text{const.}
\]  
(28)

\[
\mathcal{W} = \langle u_1 \rangle_\xi + \frac{\lambda}{2B_0} \langle \rho^{(1)} \rangle_\xi,
\]  
(29)

where the mean longitudinal hydrodynamic velocity \( \langle u_1 \rangle_\xi \) obeys the same equation as in the kinetic theory. The identity between the kinetic and MHD systems in the limit of a cold plasma is conspicuous.

Several problems can be addressed using the above kinetic version of the DNLS equation, valid for wave trains. One of them concerns the influence of the Landau damping on the filamentation phenomenon. In the Hall-MHD context, the envelope formalism predicts that filamentation of long Alfvén waves requires the condition \( \beta > 1 \). Direct evidence of this effect was demonstrated by numerical integration of three-dimensional DNLS equation including the mean fields [31], and also by comparison with direct numerical simulations of the Hall-MHD equations [20]. The extension of the envelope analysis to the kinetic DNLS system (with the mean fields) is in project and should enable one to characterize the influence of Landau damping on Alfvén wave filamentation.

Furthermore, the above system that, as already mentioned, provides an asymptotically exact fluid description for the evolution of long Alfvén waves, can also be used as a benchmark for Landau fluid models recently proposed to describe the large scale dynamics of a collisionless plasma permeated by a strong ambient field [32]. Such systems are constructed as fluid moment equations from the guiding center distribution function and can be viewed as a generalized magnetohydrodynamic description that retains ion kinetic Landau damping. In this approach, longitudinal and transverse pressure fluctuations are not assumed to adiabatically follow the density variations but obey dynamical equations that involve the heat flux and are consequently unclosed. The system is then completed by expressing the heat fluxes in terms of the lower moments and of the magnetic field perturbation. These relations are determined by matching with the linear kinetic density and perpendicular pressure response [32]. Several models of different degrees of complexity were proposed. Their respective accuracy can be evaluated by comparing their predictions for the dynamics of long Alfvén waves with the results of the exact asymptotic theory.

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REFERENCES