Halo-Formation at an Early Stage of Injection into High-Intensity Hadron Rings

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Abstract. Halo formation under a non-equilibrium state for a 2D Gaussian beam has been examined in terms of a transition of time-varying nonlinear resonances induced by the space-charge effect. Analytic approaches using an isolated resonance Hamiltonian are presented for estimating the location of the halo and the strength of the diffusion.

INTRODUCTION

One of the major issues in high-power hadron accelerators is activation of the environment surrounding an accelerator due to beam loss. Beam loss must be reduced to a sufficiently low level to allow hands-on-maintenance, so it is important to understand the mechanisms of emittance growth and halo formation, which result in beam loss.

From this point of view, halo formation has been studied by various simulations and theoretical analyses. For example, the mechanism of an injection-beam loss has been systematically studied by using flying wire monitors [1] and simulations, such as ACCSIM [1] and PATRASH [2], in the KEK 12GeV proton synchrotron (KEK-PS) since 1999, where 30% of protons are lost before acceleration. In these studies, a resonant interaction was found to be a driving mechanism of halo formation [1-3].

As is well known, an injected beam distribution into a circular accelerator suddenly varies for a short time period, which is called a non-equilibrium state in this paper, because of a mismatching of the beam distribution, resonances induced by space-charge fields, nonlinear magnets and an imperfection of magnets. Through this non-equilibrium state, the beam emittance blows up (see Fig. 1), and then a halo may be generated. Therefore, in order to understand the mechanisms for halo generation, an analysis of the beam-behavior under the non-equilibrium state is indispensable. However, because the emittance is not constant under non-equilibrium, an analysis using the envelope equation, such as the particle-core-model, cannot be applied for the non-equilibrium state. In addition, it is inaccurate to apply a simulation analysis, such as an FFT analysis or a Poincaré map analysis, for the non-equilibrium condition, because these analyses need to track over 100 turns, but the non-equilibrium state finishes in a much shorter time-period as shown in Fig.1.

The purpose of this paper is to examine halo generation under a non-equilibrium condition in a circular accelerator. In this context, useful analytic tools have been developed, which is based on an Isolated Resonance Hamiltonian (IRH) and Fokker-Planck diffusion constant. These allow one to predict the location of the halo and the strength of each time-varying resonance as a function of the beam and machine parameters, even in non-equilibrium.

Before proceeding, assumptions concerning the calculations discussed here are noted as follows. The calculations were carried out for 2-D mismatched beams with a Gaussian distribution in a typical FODO lattice. Most of the beam/machine parameters were taken from the KEK-PS, where the injection energy is 500 MeV and the circumference is 340 m. In the KEK-PS, the beam core generally oscillates 28 times per 1 turn because of the 28 FODO cells. In order to manifest a key role of the space-charge effects in halo formation, no external nonlinear magnets, no imperfection of magnets and no momentum spread...
were included in the present calculations. The combination of bare-tunes ($Q_6$, $Q_7$) chosen in the present study was close to the operational parameters, (A (7.12, 5.21) and B (7.22, 5.21)). In the case of A, a structure resonance due to a space-charge effect in the horizontal direction has been pointed out in past simulation results, but no resonance was shown in the case of B, as shown in Fig.1.

![Figure 1: Horizontal emittance growth.](image)

**ISOLATED RESONANCE HAMILTONIAN FOR A GAUSSIAN BEAM**

In the early stage of the nonequilibrium state, the whole view of the time-dependent process seems to be observed by using snapshots of the 1st-order Hamiltonian turn by turn.

The space-charge potential generated by a beam with the Gaussian distribution is

$$U(x,y;t) = -\frac{eN}{4\pi\varepsilon_0} \int_0^t \frac{1 - e^{-\left[\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right]}}{\sqrt{1 + 2\sigma_x^2} + \sqrt{1 + 2\sigma_y^2}} dt,$$

where $N$ is the total number of particles per unit length, $\varepsilon_0$ is the permittivity, $\sigma_x$ is the r.m.s. beam size and $\xi$ is $x$ or $y$, respectively. Here, action-angle variables ($\psi_x, \psi_y, I_x, I_y$) and an independent variable, $\theta = \varphi/R_0$ [4], are introduced, where $\xi = (2\beta_x^2\Delta)^{1/2}\cos(\psi_x + \phi)$, $R_0$ is the averaged orbit radius, $\beta_x$ is the Twiss parameter, and $\phi$ is the flutters of the betatron phase with respect to the averaged phase advance of the unperturbed betatron oscillation. The Hamiltonian describing the betatron oscillation perturbed by the space-charge effects is given in the form

$$H(\psi_x, \psi_y, I_x, I_y; \theta) = Q_1 I_x + Q_2 I_y$$

$$+ \frac{eR_0}{\gamma^2 \rho v} U(\psi_x, \psi_y, I_x, I_y; \theta),$$  \hspace{1cm} (2)$$

where $\gamma$, $p$, and $v$ are the relativistic mass factor, the momentum and the velocity of the on-momentum particle, respectively.

The space-charge potential can be separated into oscillating terms with the angle variable and the other oscillating terms with the dependent variable, which are originated from the fluctuation, r.m.s. beam size and Twiss parameter. The parametric nonlinear resonances between an individual particle and the intrinsic beam-core oscillation are known to be excited when the phase of a Fourier term of $U$ slowly varies with $\theta$. Because the past simulation results have shown nonlinear resonances in the horizontal direction [3], we focus on the lowest slowly oscillating phase, $2a\psi_x - b\theta$, where $a$ and $b$ are integers. The other slowly oscillating phases are given by $i(2a\psi_x - b\theta)$, where $i$ is an integer. The IRH is known to be obtained by averaging the Hamiltonian with respect to $\theta$ [5]. In this process, rapidly oscillating terms disappear. Furthermore, since $\langle H \rangle$, where $\langle \rangle$ means the averaging with respect to $\theta$, is not a constant of the motion, the canonical transformation from $\psi_x, I_x$ to $(\Psi_x = \psi_x - b\theta/(2a), I_x)$ is made. Finally, we arrive at the IRH describing the parametric nonlinear resonance between the betatron oscillation and the oscillating space-charge forces.

$$H_{\text{iso}}(\Psi_x, I_x, I_y) = \left( \frac{Q_x - b}{2a} \right) I_x$$

$$+ \frac{eR_0}{\gamma^2 \rho v} \left( U(\Psi_x, I_x, I_y) \right),$$  \hspace{1cm} (3)$$

$$\left\{ U(\Psi_x, I_x, I_y) \right\} = U_0(I_x, I_y)$$

$$+ \sum_i U_i(I_x, I_y) \sin(2\Psi_x + \Psi_i(I_x, I_y)),$$  \hspace{1cm} (4)$$

$H_{\text{iso}}$ and $I_i$ in Eq. (3) become constants of the motion. Details concerning the evaluation of Eq. (3) are given in Ref. [6].

Eq. (3) was numerically calculated as follows. $\beta_x$ and $\psi_{x,0}$ can be given as the definition when the machine parameters are decided. The information concerning $\sigma_x$ for each turns was evaluated by PATRASH, then $U$ was expanded by Fourier series for $0<\theta<2\pi$. This means that Eq (2) was time-averaged for...
one turn. The non-equilibrium state is maintained through more than 10 turns, as shown in Fig. 1, which is clearly longer than the above period. Therefore, the time-dependent process under non-equilibrium can be observed by using the averaged Hamiltonian.

FOKKER-PLANCK DIFFUSION EQUATION

The location of halo can be predicted by using IRH. While a strength of a resonance can be estimated by measuring a width of the resonance island, it seems to be hard to predict it by using IRH, because a time-varying resonance is hard to evaluate from \( H_{iso}(\Psi_x, I_x, I_y) \). Therefore, Fokker-Planck diffusion constant is introduced as a measure to numerically estimate the strength of each time-varying resonance.

According to chaos theory [7], particles in adiabatic Hamiltonian system lie on a KAM-surface, even under some perturbations. However, the particles can jump from the KAM surface, when the resonances are caused through following processes: (1) Homoclinic chaos. In the case of a low-dimensional resonance, KAM surfaces near a separatrix are destructed, and then a stochastic layer is constructed. Particles move in this layer. (2) In the case of a high-dimensional resonance, such as a coupling resonance, Arnor diffusion is caused. In this paper, the homoclinic chaos should be chosen for the one-dimensional resonances represented by Eq. (3).

As the context of existing chaos text [7], the equation to represent the beam behavior on the homoclinic chaos is derived as follows. The Vlasov equation for particles is given by

\[
\frac{\partial f(\theta)}{\partial \theta} = -L(\theta)f(\theta),
\]

\[
L(\theta) = \frac{\partial H}{\partial I_x} \frac{\partial}{\partial \Psi_x} - \frac{\partial H}{\partial \Psi_x} \frac{\partial}{\partial I_x} + \frac{\partial H}{\partial I_y} \frac{\partial}{\partial \Psi_y} - \frac{\partial H}{\partial \Psi_y} \frac{\partial}{\partial I_y},
\]

where \( f(\theta) \equiv f(\Psi_x, \Psi_y, I_x, I_y; \theta) \) is a beam distribution in phase space. Integrating Eq.(5) by \( \theta \),

\[
f(\theta) = f(0) - \int_0^\theta d\theta L(\theta')
\]

and substituting Eq. (7) into the integrand of Eq. (7),

\[
f(\theta) = f(0) - \int_0^\theta d\theta L(\theta') \times [f(0) - \int_0^{\theta'} d\theta'' L(\theta'')f(\theta'')]
\]

is given. After expanding \( f \) by Fourier series with respect to the angle variables, multiplying by \( \exp(-j(h \Psi_x + k \Psi_y)) \), where \( h \) and \( k \) are integers, integrating by the angle variables, using a random-phase approximation [7] and differentiating by \( \theta \) with some processes,

\[
\frac{\partial F_0(I_x, I_y; \theta)}{\partial \theta} = \frac{1}{2} \frac{\partial}{\partial I_x} \left[ D(I_x, I_y) \frac{\partial F_0(I_x, I_y; \theta)}{\partial I_x} \right],
\]

\[
F_0(I_x, I_y; \theta) = \frac{1}{4\pi^2} \int_0^{2\pi} d\Psi_x \int_0^{2\pi} d\Psi_y
\]

\[
\times f(\Psi_x, \Psi_y, I_x, I_y; \theta),
\]

\[
D(I_x, I_y) = \sum_i D_i(I_x, I_y),
\]

\[
D_i(I_x, I_y) = 2\pi \left\{ \frac{eN}{Y^2 \rho V} U_i(I_x, I_y) \right\}^2
\]

are achieved from Eq. (5) as the equation to represent the beam behavior on homoclinic chaos. As is well known, Eq. (9) is the Fokker-Planck diffusion equation. This means that the averaged beam distribution with respect to the angle variables near the resonant condition diffuses in the stochastic layer in the \( I_x \)-direction. Moreover, it is notable for Eq. (12) that the diffusion constant is given by the amplitude of the resonant-potential of Eq. (4). In concrete, \( D_1 \) means the diffusion constant due to a mismatching resonance represented by \( U_1 \), and \( D_2 \) represents that due to a structure resonance of \( U_2 \). Stronger \( U_i \) gives larger \( D_i \), and then particles strongly diffuse in the stochastic layer. This means that \( D_i \) can be applied for representing the strength of each time-varying resonance, which can be numerically evaluated from the \( H_{iso} \).
The diffusion constant of Eq. (12) depends on the $I_y$ as well as $I_x$, though the diffusion is caused in only the horizontal direction. As expected [2], the diffusion constant becomes smaller as the $I_y$ becomes larger (see Fig. 2), where the angle variables of the unstable fixed point is used for the evaluation because the diffusion is caused near the separatrix. Therefore, $I_y = 0$ was supposed in the next section in order to measure the maximum resonant effects.

While the diffusion may indeed occur around the unstable fixed points, this should not be dominant for the emittance growth because the system rapidly changes under the non-equilibrium. Therefore, the diffusion process is not discussed in this paper. The only diffusion constant is used for the criterion to represent the strength of the time-varying resonances in the following section.
TIME-VARYING NONLINEAR RESONANCES INDUCED BY COMBINATION OF BEAM CORE OSCILLATION

Substituting the realistic time-varying rms. beam size under non-equilibrium, which was evaluated from simulation results [7], into Eq. (3) and (12), the time-varying nonlinear resonances in the cases of A and B are examined for the early ten turns. By comparing them of each turn, the phenomena at the early stage of injection can be clearly seen.

(Case A) $D_1$ and $D_2$ are shown in Fig. 3(a), and the phase-space structures are shown in Fig. 4. As shown in Fig. 3(a), $D_2$ is slightly larger than $D_1$ during the early few turns. This means that the emittance growth is caused by both the structure resonance and the mismatching resonance, and that two resonance islands caused by the mismatching resonance seem to be dominant in the phase space (see Figs. 4(a) and 4(b)) at this time, in which the mismatching component remains in the beam-core oscillation induced by the FODO lattice of the KEK-PS. Furthermore, Figs. 3(a), 4(c) and 4(d) indicate that the nonlinear resonance is switched to the structure resonance after the 4th turn, where the mismatching component is lost because of the growth of filamentation [7]. Because $D_2$ is still large there, the halo tends to grow.

(Case B) $D_1$ and $D_2$ are shown in Fig. 3(b), and the phase-space structures are presented in Fig. 5. $D_2$ is negligible through 10 turns. This means that the resonance caused by mismatching is dominant at the early stage. Moreover, $D_1$ for case B is 1/10 smaller than that for case A. This suggests that the emittance growth for case B is smaller than that for case A. In addition, the nonlinear resonance is rapidly lost after decay of the mismatching as well as case B (see Figs. 3(b), 5(c) and 5(d)). Because there are no resonances, the beam distribution is smeared out due to the nonlinear space-charge fields. Then, the beam distribution may arrive at the equilibrium.

Those descriptions suggested by $H_{iso}$, $D_1$ and $D_2$ are consistent with the simulation result shown in Fig. 1.

CONCLUSION

In order to understand the mechanisms for the halo generation, an analysis for the beam-behavior under the non-equilibrium is indispensable. From this point of view, analytic tools based on the isolated resonance Hamiltonian and Fokker-Planck diffusion constant have been developed. These equations have been proved to be a useful tool to estimate the location of the halo and the strength of each time-varying resonance in order to understand the whole story at the early stage of injection; what phenomena precisely happen in the process, and what mechanism can drive these phenomena. It has been concluded that the halo is driven by a time-varying nonlinear resonance excited by the intrinsic beam core oscillation at the non-equilibrium state. In addition, the beam distribution arrives at an equilibrium state through the decay process of the nonlinear resonances.

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