Space Charge Effects in Bunches for Different rf Wave Forms

Oliver Boine-Frankenheim* and Tripti Shukla†

*Gesellschaft für Schwerionenforschung mbH, Planckstr.1, 64291 Darmstadt, Germany
†Indian Institute of Technology (IIT), New Dehli-110016, India

Abstract. As part of the required beam stability and feedback studies for the SIS upgrade we investigate the interplay of nonlinear rf fields and space charge using simplified analytic models as well as large scale particle simulation scans. Starting from the matched elliptic (‘Hofmann-Pedersen’) distribution analytic expressions for the synchrotron tune and for the rigid dipole mode are obtained. The threshold intensities for the space charge induced loss of Landau damping in single and double rf wave forms are derived. The thresholds are compared with machine observations and with simulations of the bunch response to a weak rf phase modulation. The simulation results are related to previous work on beam transfer functions in single and double rf waves.

INTRODUCTION

Longitudinal space charge effects play an important role in storage rings or synchrotrons for high current ion beams. The induced effects range from synchrotron tune shifts (see e.g. [1]) to coherent mode splitting [2] that can both be observed with high accuracy from the Schottky noise spectrum, as demonstrated in the GSI heavy ion cooler storage ring ESR [3].

Below transition space charge reduces the effective rf voltage seen by the beam particles. This usually requires an increase of the applied rf voltage in order to compensate for the reduction of the bucket area. Space charge affects the frequencies and the damping of coherent bunch modes. This in turn changes the bunch instability thresholds and the corresponding impedance budget.

For bunches that are very short relative to the rf wave length, but still long compared to the beam pipe diameter it is straightforward to calculate the space charge induced incoherent synchrotron frequency shift (see e.g. [1]). The space charge induced coherent mode splitting in short bunches was analyzed by Neuffer in Ref. [2].

At low or medium beam energies bunches usually cannot be regarded as being short relative to the rf wave length. Particles with large amplitudes will be affected by the nonlinear components of the rf field and by space charge. This is especially this case if a second harmonic rf system is employed in order to flatten the bunch profile and to increase the transverse space charge limit (see e.g. [4]). In case of such a double rf system the particle motion is fully nonlinear. A double rf system is foreseen in order to increase the bucket area and also the transverse space charge limit in the SIS [5]. Because of the low injection energy (11.4 MeV/u) and the demand for highest longitudinal beam quality the intense Uranium bunches will be strongly affected by longitudinal space charge.

An excellent review of nonlinear single particle motion in single and double rf waves forms can be found in Ref. [6]. A self-consistent treatment of matched bunches affected by space charge in arbitrary rf wave forms was presented by Hofmann and Pedersen [7] and applied to long bunches in a single rf wave. They obtained an analytic expression for the threshold intensity for the ‘loss of Landau damping’ in case of the rigid dipole mode in a single rf wave. For sufficiently high intensities space charge suppresses the decoherence of dipole oscillations, leading to persistent oscillations of intense bunches [8, 9]. In the present work we apply the theory by Hofmann and Pedersen to double rf systems.

Of particular interest for the study of beam stability thresholds is the response of long bunches to rf phase or amplitude modulations. In Ref. [10] the response of a bunch to small rf phase or amplitude modulations was studied in the framework of the beam transfer function formalism. It was found that long bunches in a double rf wave are intrinsically unstable, because of the vanishing derivative of the synchrotron frequency inside the bunch (but outside the bunch center). In a system with non monotonic behavior of the synchrotron frequency Landau damping can be lost for certain particle amplitudes. Whether the infinite response function obtained in Ref. [10] from linearized Vlasov theory lead to observable effects in high resolution, non-perturbative simulation studies is one of the subjects of the present work.
LONGITUDINAL EQUATION OF MOTION

Let $\phi$ be the phase coordinates of an off-momentum particle. Then the longitudinal equation of motion is

$$\dot{\phi} = \frac{1}{m} \frac{n q V}{L} \tag{1}$$

with the effective mass $m^* = -\gamma_0 m / \eta$, the relativistic parameter $\gamma_0$, the slip factor $\eta$, the ring circumference $L$ and radius $R$, the charge $q$, the harmonic number $n$ and the voltage profile $V(\phi)$. The voltage profile for a single ($\alpha = 0$) and for a double ($\alpha > 0$) rf system, operating at the second harmonic of the main rf, is given through (see e.g. [6]):

$$V(\phi) = \frac{V_0}{2} \sin(\phi) - \alpha (\sin(\phi_2) + 2(\phi - \phi_2) - \sin(\phi_2)) \tag{2}$$

with the synchronous phases $\phi_1$ of the main rf and of the second rf $\phi_2$, respectively. In order to obtain a flattened rf potential well with a double rf system the first and the second derivative of the voltage profile should vanish at $\phi = \phi_1$, [4]. For stationary bunches one obtains $\alpha = 0.5$. The equation of motion in the $(\phi, v)$ coordinates can be derived from the 'Hamiltonian'

$$H = \frac{\dot{\phi}^2}{2} - \omega_\phi^2 Y(\phi) \tag{3}$$

with the potential

$$Y(\phi) = \frac{1}{V_0} \int_{\phi_1}^{\phi} V d\phi, \tag{4}$$

and the small amplitude synchrotron frequency for $\alpha = 0$ and $\phi = 0$

$$\omega_\phi^2 = \frac{2q CV_0}{RE_0^2 m^*} \tag{5}$$

The voltage profile can be divided into the external (rf) voltage part and the space charge part

$$V = V_{rf} + V_s \tag{6}$$

The space charge voltage is given through (see e.g. Ref. [1])

$$V_s = 2\pi R E_s = -q \beta_e c R X_s \frac{n^2}{R^2} \frac{\partial \lambda}{\partial \phi} \tag{7}$$

$$X_s = \left| \frac{Z_c}{n} \right| = \frac{g}{2e_0 \beta_0 c^2 \gamma_0^2} \tag{8}$$

with the space charge electric field $E_s$, the space charge reactance $X_s$, the line density $\lambda(\phi)$ and the g-factor. For the space charge potential one obtains

$$Y_s = q \beta_0 c R X_s \frac{n^2}{V_0} (\lambda_0 - \lambda) \tag{9}$$

with the line density $\lambda_0$ at $\phi = \phi_s$.

ELLiptic BUNCH DISTRIBUTION

If the Hamiltonian is a constant of motion any stationary ('matched') distribution function can be written as a function of $H$. The analytic analysis in the presence of space charge can be greatly simplified if a local elliptic ('Hofmann-Pedersen') distribution function [7] is assumed

$$g(H) = c_1 \sqrt{H - H_m} \tag{10}$$

$$H_m = \frac{v_m^2}{2} = -\alpha_0^2 Y(\phi_m) \tag{11}$$

with the normalization constant $c_1$, the value of the Hamiltonian $H_m$ for the bunch boundary particle, the maximum phase velocity $v_m = \phi_m$ at the bunch center $\phi = \phi_m$, and the potential $Y(\phi_m)$ at one end of the bunch. For the distribution function in $(\phi, v)$ space we get

$$f(\phi, v) = c_2 \sqrt{v_m^2(\phi) - v^2} \tag{12}$$

with the phase velocity function for the boundary particle

$$v_m^2(\phi) = \alpha_0^2 Y(\phi) - Y(\phi_m) \tag{13}$$

The line density is obtained from Eq. 12 as

$$\lambda(\phi) = c_3 (Y(\phi) - Y(\phi_m)) = \frac{N}{u_m} (Y_{rf}(\phi) - Y_{rf}(\phi_m)) \tag{14}$$

$$u_m = \int_{\phi_m}^{\phi_m} (Y_{rf}(\phi) - Y_{rf}(\phi_m)) d\phi \tag{15}$$

with the number of particles $N$ in the bunch. In the case of an elliptic distribution function the space charge potential induced by a bunch is directly proportional to the external potential. The total potential can be written in the form

$$Y(\phi) = Y_{rf}(\phi) \left( 1 - \frac{V_{rf}}{V_0} \right) \tag{16}$$

$$V_{rf} = \text{sgn}(m^*) q \beta_0 c R X_s \frac{n^2}{R^2} \frac{N}{u_m} \tag{17}$$

with the space charge voltage amplitude $V_{rf}$. Below transition ($m^* > 0$) the limiting bunch intensity is given through $V_{rf} = V_0$. At this intensity the external focusing field is exactly canceled by the space charge field. Below this limiting bunch intensity and for bunch boundaries $\phi_{m1}, \phi_{m2}$ not exceeding the bucket boundaries the matched voltage amplitude can be obtained as

$$\alpha_0^2 = -\frac{v_m^2}{2Y_{rf}(\phi_m)}(1 + \Sigma) \tag{18}$$

with the space charge parameter

$$\Sigma = \frac{1}{V_0/V_{rf} - 1} \tag{19}$$
SYNCHROTRON FREQUENCY

The synchrotron period as a function of the left and right particle oscillation amplitudes $\phi_1, \phi_2$ can be derived from

$$T_s(\tilde{\phi}_2) = \frac{2 R}{n} \int_{\phi_1}^{\phi_2} d\phi \frac{1}{\sin(\phi)}$$  \hspace{1cm} (20)

with the velocity amplitude function

$$\hat{v}^2(\phi) = \frac{q V_0}{\pi n m^*} (Y(\phi) - Y(\phi_2))$$  \hspace{1cm} (21)

For the elliptic distribution the effect of space charge can be cast into a simple, multiplicative factor

$$T_s = \frac{\sqrt{2}}{\omega_0} \sqrt{1 + \frac{\pi}{1 + \Sigma}} \frac{\int_{\phi_1}^{\phi_2} d\phi}{\sqrt{|Y_f(\phi) - Y_f(\phi_2)|}}$$  \hspace{1cm} (22)

In the case of a single, stationary rf wave and small amplitudes ($\hat{\phi} \ll \pi$) the following result for the synchrotron oscillation frequency can be obtained (see e.g. [6], p. 235, for $\Sigma = 0$)

$$\frac{\omega_s}{\omega_0} \approx \sqrt{1 + \frac{\phi^2}{16}}$$  \hspace{1cm} (23)

For a double rf wave we obtain

$$\frac{\omega_s}{\omega_0} \approx \sqrt{1 + \frac{\pi}{1 + \Sigma \frac{23}{2} K(1/\sqrt{2})}}$$  \hspace{1cm} (24)

with the elliptic integral of the first kind $K(x)$. In a stationary double rf wave the maximum synchrotron frequency is located at $\hat{\phi}_{\text{crit}} \approx 117^\circ$ with the numerical value given by $\omega_s^{\text{max}} \approx 0.78 \omega_0 / \sqrt{1 + \Sigma}$. In Ref. [10] it was shown that if the bunch length $\phi_m$ exceeds $\hat{\phi}_{\text{crit}}$ Landau damping will be lost for frequencies close to $\omega_s^{\text{max}}$ because of the vanishing derivative of the synchrotron frequency. Therefore this amplitude and the corresponding synchrotron frequency are called 'critical'.

RIGID DIPOLE OSCILLATIONS

Let $\phi_c$ be the position of the bunch center. If a matched bunched is rigidly displaced by the amount $\Delta \phi_c = \phi_c - \phi_s$ from the synchronous phase the net force acting on the bunch center can be obtained from the rf force averaged over the bunch profile (see also Ref. [7])

$$\hat{\phi}_c = \hat{\phi}_c \frac{1}{N} \int_{\phi_m}^{\phi_n} \frac{V_{rf}}{V_0} \lambda(\phi - \phi_c) d\phi$$  \hspace{1cm} (25)

Expanding the integrand for small displacements ($\Delta \phi_c \ll \pi$) yields the equation of motion for a harmonic oscillator with the dipole oscillation frequency

$$\Omega_c^2 = \omega_0^2 \left(\frac{V_{rf}}{V_0}\right)^2 \frac{1}{|\hat{\phi}_c|} \int_{\phi_m}^{\phi_n} \lambda(\phi - \phi_c) d\phi$$  \hspace{1cm} (26)

For a stationary single rf wave we obtain

$$\left(\frac{\Omega_c}{\omega_0}\right)^2 = \frac{2 \phi_m - \sin 2 \phi_m}{4 \sin \phi_m - 4 \phi_m \cos \phi_m}$$  \hspace{1cm} (27)

and for a double rf wave

$$\left(\frac{\Omega_c}{\omega_0}\right)^2 = \frac{5 \phi_m - \sin \phi_m - \frac{1}{2} \sin 2 \phi_m - \frac{1}{15} \sin 4 \phi_m + \frac{1}{2} \sin 3 \phi_m - 2 \phi_m \cos \phi_m + \frac{1}{2} \phi_m \cos 2 \phi_m + 2 \sin \phi_m - \frac{1}{3} \sin 2 \phi_m}{-2 \phi_m \cos \phi_m + \frac{1}{2} \phi_m \cos 2 \phi_m + 2 \sin \phi_m - \frac{1}{3} \sin 2 \phi_m}$$

In the limit of short bunches one obtains for a single rf wave

$$\left(\frac{\Omega_c}{\omega_0}\right)^2 = \left(1 - \frac{\phi_m^2}{10}\right)^{1/2}$$  \hspace{1cm} (29)

and for a double rf wave

$$\left(\frac{\Omega_c}{\omega_0}\right)^2 = \frac{5}{14} \phi_m$$  \hspace{1cm} (30)

For a single rf wave the dipole mode as well as the incoherent synchrotron frequency can be identified from the simulation 'Schottky' noise from a Particle-In-Cell (PIC) code [11]. Fig. 1 shows the simulation noise from a matched elliptic distribution of macro-particles as a function of the relative frequency $\Delta \omega = \omega - h \omega_0$ divided by $\omega_0$ at harmonic $h = 10$ for $\Sigma = 1.0$ and $\phi_m = 90^\circ$. In case of a double rf wave the noise spectrum remains incoherent also in the presence of space charge.

![FIGURE 1. Simulation noise power spectrum at harmonic $h = 10$ from a matched bunch in a single rf wave. The space charge parameter is $\Sigma = 1$ and the bunch length $\phi_m = 90^\circ$.](image-url)
LANDAU DAMPING OF DIPOLE OSCILLATIONS

Landau damping is lost when the coherent frequency (here the dipole mode) is outside the band of incoherent synchrotron frequencies. The corresponding threshold intensity can be calculated from

\[ \Omega_c = \omega_s^{\min}, \quad \text{or} \quad \Omega_c = \omega_s^{\max} \]  

(31)

with \( \omega_s^{\min} \) (\( \omega_s^{\max} \)) being the minimum (maximum) synchrotron frequency inside the bunch. The first case is e.g. relevant for a single rf wave above transition (or in the case of a broadband inductive impedance below transition), where the single particle synchrotron frequencies are shifted upwards with increasing space charge. Here the threshold intensity is determined by the lowest synchrotron frequency. Below transition (or for a capacitive impedance above transition) the second case applies.

For short bunches the threshold parameters can be obtained analytically. In a single rf wave we obtain with \( \omega_s^{\max} = \omega_s \sqrt{1 + \Sigma} \) for \( m^* > 0 \)

\[ \Sigma_{\text{th}} = \frac{1}{\sqrt{1 - \frac{\omega_s^{\max}}{\omega_c}}} - 1 \approx \frac{\phi_m^2}{10} \]  

(32)

and with \( \omega_s^{\min} = \omega_s(\phi_m)/\sqrt{1 - |\Sigma|} \) for \( m^* < 0 \)

\[ |\Sigma_{\text{th}}| = \frac{\phi_m^2}{8} \]  

(33)

In a double rf wave \( \omega_s^{\min} = 0 \) holds. The band of incoherent synchrotron frequencies in a double rf wave extends from 0 to \( \omega_s^{\max} \). For positive \( m^* \) and for a short bunch one obtains the constant threshold space charge parameter

\[ \Sigma_{\text{th}} = \frac{7}{20} \frac{\pi^2}{k^2(1/\sqrt{2})} - 1 \approx 0.005 \]  

(34)

It is interesting to point out that for negative \( m^* \) the criterium for Landau damping Eq. 31 is always fulfilled in case of a double rf wave. In Fig. 2 \( \Sigma_{\text{th}} \) is shown for stationary single and double rf waves. It can be seen that in a single rf wave with increasing bunch length the threshold space charge parameter increases \( \sim \phi_m^2 \). In a double rf wave Landau damping is lost below transition at much lower space charge parameters than in a single rf wave. Slightly above the critical bunch length \( \phi_{\text{crit}} \approx 117^0 \) Landau damping is lost for all \( \Sigma \). For bunch lengths exceeding this value \( \Sigma_{\text{th}} \) increases again. Persistent dipole oscillations of ion bunches in single rf waves with \( \Sigma > \Sigma_{\text{th}} \) can be observed in the GSI heavy ion synchrotron SIS [9]. This is due to the present lack of an rf phase control system in the SIS. If the energy of the injected beam from the UNILAC linear accelerator differs from the rf cavity frequency the coasting beam is captured with a momentum offset.

**BUNCH RESPONSE TO A RF PHASE MODULATION**

The intensity threshold for the loss of Landau damping was obtained in a non-self-consistent fashion. A more rigorous analytical approach would start from the linearized Vlasov theory. For \( \Sigma = 0 \) this approach was pursued in Ref. [10] where the beam transfer functions (BTF) in single and double rf waves were calculated for rf phase modulations (in a double rf wave only the phase of the first rf wave is modulated). The BTF amplitudes for single and double rf waves show pronounced maxima at \( \Omega = \Omega_c \), with \( \Omega_c \) being the rigid dipole oscillation frequency. In a double rf wave and for bunch lengths equal to or longer than \( \phi_{\text{crit}} \) the BTF amplitude diverges for \( \Omega = \omega_s^{\max} \). The infinite response for modulation frequencies \( \Omega = \omega_s^{\max} = \Omega_{\text{crit}} \) is due to the vanishing derivative of the synchrotron frequency inside the bunch. For those frequencies Landau damping is lost and within a linearized theory there is no other damping mechanism.

In the present work we study the bunch response to a weak rf phase modulation within a PIC code starting from a matched elliptic distribution. The maximum dipole amplitudes are excited for modulation frequencies tuned close to the rigid dipole mode. For bunch intensities exceeding the threshold space charge parameter for

![Figure 2](image-url)
the loss of Landau damping a strong increase in the maximum dipole amplitudes can be observed in both, single and double rf waves. For long bunches, exceeding the critical bunch length \( \phi_{\text{crit}} \approx 117^0 \) in a double rf wave, we do not observe a pronounced dipole response for \( \Omega = \omega_{\text{crit}} \), as it would be expected from the BTF. Instead characteristic bunch shoulders around \( \phi = \pm \phi_{\text{crit}} \) (see Fig. 3) are formed. It is interesting to note that similar shoulders on long bunches in a double rf wave were measured in the CERN SPS [12]. How these bunch shoulders affect and eventually restore Landau damping will be a topic of future work.

CONCLUSIONS

In the framework of the elliptic distribution function we obtained analytic expressions for the synchrotron frequency, the rigid dipole oscillation frequency and for the threshold space charge parameter for the loss of Landau damping in single and double rf wave forms. It was shown that below transition energy the threshold in a double rf wave is much lower than in a single rf wave. Above transition Landau damping is always effective in a double rf wave. We showed that the synchotron frequency and the rigid dipole mode frequency can both be well identified from the ‘Schottky’ simulation noise from long bunches in a single rf wave. The bunch response to a weak rf phase modulation obtained from PIC simulations shows the maximum dipole amplitudes are excited for modulation frequencies tuned close to the rigid dipole mode. For bunch intensities exceeding the threshold space charge parameter \( \Sigma_{\text{th}} \) for the dipole mode accurately predicts the observed strong increase of the bunch response in single and double rf waves. Preliminary simulation studies including resistive impedances sources show that \( \Sigma_{\text{th}}(\phi_m) \) can approximate very well the instability thresholds in long bunches affected by space charge. Future work will also address the effect of space charge on high order modes in long bunches. Especially the quadrupolar and sextupolar modes will be studied. The experience at the PSB e.g. shows that especially sextupolar modes can be excited in double rf waves [13].

ACKNOWLEDGMENTS

The authors would like to thank E. Shaposhnikova (CERN) and H. Damerau (CERN) for relevant remarks.

REFERENCES

8. A. N. Lebedev, Atomnaya Energiya 25, 100 (1968)
12. E. Shaposhnikova, private communication, 2004