Analytic Differentiation of Barlat’s 2D Criteria for Inverse Modeling

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Abstract. The demand for alternative identification schemes for identification of constitutive parameters is getting more pronounced as the complexity of the constitutive equations increases, i.e. the number of parameters subject to identification. A general framework for inverse identification of constitutive parameters associated with sheet metal forming is proposed in the article. The inverse problem is solved, through minimization of the least square error between an experimental punch force sampled from a deep drawing and a predicted punch force produced from a coherent finite element model.

Keywords: least square formulation, trust region, totally structured secant methods, finite element, inverse modeling, constitutive parameters, anisotropic material, plastic deformation.

INTRODUCTION

The demand for alternative identification schemes for identification of constitutive parameters is getting more pronounced as the complexity of the constitutive equations increases, i.e. the number of parameters subject to the identification, see e.g. [1, 2], where 8 parameters are used to describe the yield locus and additional parameters are used for modeling the hardening behavior. The material cannot be modeled exclusively from uni-axial tensile tests which presently are the dominating material test; biaxial tension test or texture information are needed to fit the constitutive parameters. Moreover, the uni-axial tensile test only covers a limited strain and stress range, which is significantly lower than the deformations range expected in e.g. a deep drawing operation, the constitutive framework is therefore based on extrapolation.

A general framework for inverse identification of constitutive parameters associated with sheet metal forming is proposed. The inverse problem is solved, through minimization of the least square error between an experimental punch force \( F_{em} \), sampled from a deep drawing, and a predicted punch force \( F_{fit} \) produced from a coherent finite element model, the object function is defined as:

\[
f(x) = \frac{1}{2} \sum_{j=1}^{m} (F_{j}^{fit} - F_{j}^{em})^2.
\]

The Jacobian matrix was defined directly in the material routine, hence is was defined analytically through differentiation of the plastic force \( F_{plast} \) eq. (15), with respect to the constitutive parameters, i.e. hardening \( (\sigma_{eq} = K \varepsilon_{eq}^{n}) \) and the anisotropy behavior was modeled by Barlat’s 2D criteria \( (\sigma_{eq} = \Phi(a, h, p, M)) \).

A nonlinear least square algorithm was applied for the minimization scheme, and the object function was minimized by altering the constitutive parameters, defined by an exponential hardening law and Barlat’s 2-D yield criteria [3]. Moreover the friction coefficients between the sheet-metal and tool parts were also introduced as design parameters.

The proposed inverse modeling scheme was tested on data produced from a square deep-drawn cup, where the punch-force was sampled during the drawing, the tool dimensions correspond to the Benchmark tool for the Numisheet’93 conference, see figure 1. The identified constitutive parameters were compared with parameters determined from a uni-axial tensile test.

OPTIMIZATION SCHEME

The optimization code was originally developed for inverse modeling[4, 5, 6, 7], i.e. identification of constitutive parameters, through minimization of an object function \( f(x) \), which describes the error between the response sampled from the experiment and the simulated response

\[
f(x) = \frac{1}{2} r(x)^T r(x)
\]
where \( x \) and \( r(x) \) represent the constitutive parameters and the residual vector, respectively.

The object function was approximated by a quadratic function defined as
\[
\psi(s_{k+1}) = \nabla f(x_k)^T s_k + \frac{1}{2} s_k H s_k^T \tag{2}
\]
\[
f(x_k + s_k) = f(x_k) + \psi(s_k) \tag{3}
\]
\[
\nabla f(x) = J(x)^T r(x) \tag{4}
\]
\[
H = J(x)^T J(x) + \sum_{i=1}^{m} r_i(x) G_i \tag{5}
\]
where \( \nabla f(x) \) and \( H \) represent the gradient and the Hessian matrix respectively, \( G_i \) is a symmetric \( n \times n \) matrix containing the derivatives of \( r_i(x) \). If the residuals or second derivatives are small, then the second order part of the Hessian matrix approaches, zero and can be neglected, this approach is known as the Gauss-Newton method. However, the method may perform poorly when the residuals are nonzero in the solution, or when the object function is highly nonlinear, [8]. Within the current setup, i.e. process optimization, a nonzero solution is expected. The second order term was therefore approximated as \( \sum_{i=1}^{m} r_i(x) G_i \simeq A_k \) by the Totally Structured Secant Method, proposed by Huschens [9, 10], additional the Huschens scaling scheme can be combined with the projected scaling scheme [11, 12]. For more informations on performance of the three different least-square formulations, see [13]. TSSM and the projected scaling scheme are members of the convex Broyden Class and the SR1-update was used for computing the Broyden factor \( \phi \), see e.g. [14, 11].

The step size \( s_k \) is computed by solving the trust region subproblem,
\[
\min_{s \in \mathbb{R}^n} = \nabla f(x_k)^T s_k + \frac{1}{2} s_k H s_k^T, \quad \|s_k\| \leq \Delta_k \tag{6}
\]
where \( \Delta_k \) is the radius of the trust region. The trust region subproblem was solved by applying the Cholesky factorization scheme proposed by Sorensen [15, 16]. The method is well suited for small problems \( n < 500-1000 \), and produces a nearly exact solution to the subproblem eq. (6).

Box constraints were utilized on the parameters. The implemented method is proposed by Coleman et al. and works within the trust region framework, the method is based on the affine scaling method used in linear programming, where the box constraints are introduced into the quadratic model, eq. (2) by the definition of two scaling matrices, which yield a constrained quadratic model [17, 18]:
\[
\hat{\psi}(s_k) = (s_{k+1}) = D_k \nabla f(x_k)^T s_k + \frac{1}{2} s_k (D_k (H + C_k) D_k) s_k^T \tag{7}
\]
\[
s_k+1 = D_k s_{k+1} \tag{8}
\]
The scaling matrices \( D_k \) and \( C_k \) constrain the solution space for the model quadratic \( \hat{\psi}(s_k) \), within the bounds \( l_i < x_i < u_i \) where \( l \) and \( u \) represent the lower and upper bounds. The scheme does, however, not guarantee a strictly feasible solution. Heinkenschloss et al. propose an additional method to enforcing feasibility, by projecting the step onto the feasible domain, which significantly improves the performance, compared with the original proposal by Coleman et al. [19]. The main benefit from this approach is that the bound constrained problem is reduced to the solution of a quadratic model
\[
\min_{x \in \mathbb{R}^n} \hat{\psi}(s_k) \quad s_k \leq \Delta_k.
\]
Poor scaling of the problem can lead to poor numerical performance, a strategy is to reformulate the quadratic model by introduction of a scaling matrix \( S^x \) defined as
\[
S^x_k = diag(a_{x_k}) \tag{9}
\]
\[
\psi^x(w_k) = (S^x_k \nabla f(x_k))^T w_k + \frac{1}{2} w_k (S^x_k H S^x_k w_k)^T \tag{10}
\]
\[
x_{k+1} = x_k + S^x_k w_k \tag{11}
\]
The scaling matrix \( S^x_k \) was updated for each iteration, i.e. the diagonal elements corresponded to the current parameter set \( x_k \) [16](p. 162-166).

CONSTITUTIVE MODEL

The anisotropic behavior was modeled using Barlat’s 2D criterion [3], defined as:
\[
\Phi = a_1 |K_1 - K_2|^M + a_2 |K_1 + K_2|^M \tag{12}
\]
where the coefficients \( K_1 \) and \( K_2 \) is defined as:
\[
K_1 = \frac{\sigma_{11} + b \sigma_{22}}{2} - p^2 \sigma_{12}^2
\]
\[
K_2 = \sqrt{\left(\frac{\sigma_{11} + b \sigma_{22}}{2}\right)^2 + p^2 \sigma_{12}^2}
\]
By choosing a high value of the parameter \( M \) a yield criterion almost identical to Tresca’s is obtained. Also von Mises’ yield criterion can be obtained by choosing the isotropic version and \( M \) equal to 2, finally the Hill’s 48 criterion is obtained for \( M \) equal to 2.

The hardening behavior was modeled using an exponential hardening law \( \sigma_{eq} = K \varepsilon_{eq}^p \).
ANALYTIC GRADIENT

The punch force as a function of punch displacement reflects the energy input to the system. The basic idea is that two identical punch forces can only be obtained with the same set of constitutive parameters. The constitutive parameters were identified, by iteratively minimizing the objective function, eq. (1), where the residual vector was defined as \( r_j(x) = (F_{j}^{fit} - F_{j}^{em}) \) where \( F^{em} \) represents the empirical punch force, and \( F^{fit} \) represents the fitted data and was produced by LS-Dyna.

The objective was to minimize \( f(x) \). The minimum was identified through gradient based optimization techniques, thus an analytical expression, representing the Jacobian matrix which holds the derivatives \( \frac{\partial f(x)}{\partial x} \) is needed. Barlat’s 2D yield criterion \( \Phi(a, h, p, M) \) describes the relation between the equivalent stress \( \sigma_{eq} \) and Lankford’s coefficients which can be expressed through the parameters \( a, h, p \) and \( M \). Further, a similar relation exists between the equivalent stress and the exponential hardening law, described by the strength coefficient \( K \) and hardening coefficient \( n \).

The following scheme was applied to establish the relation between \( \sigma_{eq} \) and \( F^{fit} \). The total punch force from the model \( F^{fit} \) is defined as:

\[
F^{fit} = F^{plast} + F^{fric}
\]

where \( F^{plast} \) denotes the plastic force, i.e. the contribution to the total punch force from plastic deformation of the blank. The contribution from friction is denoted \( F^{fric} \). Independency between \( F^{plast} \) and \( F^{fric} \) as assumed. The incremental plastic work \( dw \) for one element in the finite element model can be expressed as:

\[
dw = \sigma_{eq} \Delta e_{eq}
\]

The total plastic work increment was calculated by summation of the contribution for each element. The plastic force \( F^{plast} \) can now be stated as:

\[
F^{plast} = \sum_{i=1}^{n_e} V_i \frac{\sigma_{eq,i} \Delta e_{eq,i}}{\Delta s}
\]

where \( n_e \) represents the number of elements for the blank, \( V_i \) represents the element volume and \( \Delta s \) is the increment for the punch displacement \( \Delta s \).

Under the assumption that \( F^{plast} \) is independent of the friction coefficient, the Jacobian matrix can now be defined as:

\[
J = \left[ \frac{\partial r_j}{\partial x_i} \right]_{j=1,2,\ldots,m \atop i=1,2,\ldots,n}
\]

through partial differentiation of the plastic force eq. (15), the full Jacobian matrix for the hardening and anisotropic behavior can be stated as:

\[
J = \begin{bmatrix}
1 \sum_{k=1}^{n_{pl}} V_{ak} \frac{\partial \sigma_{eq,k}}{\partial K} & \cdots & 1 \sum_{k=1}^{n_{pl}} V_{ak} \frac{\partial \sigma_{eq,k}}{\partial M} \\
1 \sum_{k=1}^{n_{pl}} V_{ak} \frac{\partial \sigma_{eq,k}}{\partial h} & \cdots & 1 \sum_{k=1}^{n_{pl}} V_{ak} \frac{\partial \sigma_{eq,k}}{\partial p} \\
\vdots & \ddots & \vdots \\
1 \sum_{k=1}^{n_{pl}} V_{ak} \frac{\partial \sigma_{eq,k}}{\partial h} & \cdots & 1 \sum_{k=1}^{n_{pl}} V_{ak} \frac{\partial \sigma_{eq,k}}{\partial p}
\end{bmatrix}
\]

The Jacobian is a \((m \times n)\) matrix, where \( n \) is the number of constitutive parameters, \( m \) is the number of points in the residual vector. The derivatives of the residual vector \( (r_i(x) = F_i^{fit} - F_i^{em}) \) with respect to \( K, n, a, h, p \) and \( M \) was assumed to be independent of the friction coefficients.

The friction is modeled using Coulomb’s friction law \( F^{fric} = \mu F^{N} \) and the sensitivity for the friction coefficient was evaluated analytically using a very simple strategy. Where the normal force is approximated using the following relation:

\[
F^{N} \approx \frac{F^{fit} - F^{plast}}{\mu}
\]

using the above approximation of the normal force the sensitivity for the friction coefficient can be defined as:

\[
\frac{\partial F^{fric}}{\partial \mu} = F^{N}
\]

assuming equal friction coefficients for all the contact interfaces between the tool parts and the sheet.

| TABLE 1. Comparison between analytical defined friction sensitivity and friction sensitivity approximated by finite difference, using a finite difference increment \( \delta \mu = 0.001 \). |
|---------------------------------|----------------|----------------|
| Finite difference | Analytic |
| \( \frac{\partial F^{fric}}{\partial \mu} \) | -595.1 | -618.4 |

For comparison, between the proposed analytical friction sensitivity scheme and the finite difference approximation, see table 1, where a increment \( \delta \mu = 0.001 \) was used for the finite difference approximation. In conclusion, the difference between the two sensitivity schemes was insignificant, thus, the simple analytical definition of the friction sensitivity seems valid.

IMPLEMENTATION

The analytic sensitivity scheme was implemented as an user defined material routine in ls-dyna version 970 double precision, the additional calculation steps compared to a normal material routine are summarized in table 2. The material routine is based on the implementation by K.B. Nielsen [20].
anisotropy behavior was modeled using Barlat’s 2D criterion. The thickness of 0.75 mm and a diameter of 160 mm. A 1368 Belytschko-Tsay shell elements with 7 integration points was used and the blank.

The inverse problem was solved in 10 iterations involving 19 LS-Dyna simulations. The result is presented in tables 3 and 4. The identified parameters is recalculated through uni-axial tests is presented in table 5.

A fixed value of the Barlat’s coefficient $M = 6.0$ was used and the robustness of the proposed scheme was tested using two initial guesses on the parameters $x_0 = [\mu \ K \ n \ a \ h \ p]$. The identified constitutive parameters are compared with the parameters from a uniaxial tensile test.

The following solution scheme was applied:

- Analytic definition of the Jacobian with respect to hardening, anisotropic behavior and friction.
- Numerical noise, present in the simulated punch force, was minimized using a second-order Butterworth low-pass filter. The numerical noise has a significant impact on the quality of the gradients and the performance of the inverse solver [7, 4].
- The second order derivative was approximated using the Projected Scaling scheme [11, 12].
- The trust region subproblem was solved using an iterative solver [15].
- Due to poor scaling of the inverse problem, an elliptic trust region scaling scheme was applied [4]. The proposed inverse problem cannot be solved without proper scaling.
- Bounds on the parameters were introduced with in the trust region framework, the method was proposed by Coleman et al. [17, 18]. Bounds on the solution space stabilize the solution and without proper bounds the inverse problem will not converge to a stable solution.

**RESULTS**

The inverse problem was solved in 10 iterations involving 19 LS-Dyna simulations. The result is presented in tables 3 and 4. The identified parameters is recalculated through uni-axial tests is presented in table 5.

A nearly perfect fit between the experimental and simulated punch force was achieved, figures 4 and 5, where

**EXPERIMENTAL AND NUMERICAL SETUP**

The process was simulated with the explicit code LS-Dyna version 970. The process time was scaled to 20 milliseconds and a forced time step $\delta t = 2.0 \ 10^{-7}$ was applied, utilizing mass scaling to improve computational efficiency [21]. The blank was modeled using 1368 Belytschko-Tsay shell elements with 7 integration points through the thickness [22]. Due to symmetry only a quarter of the cup was modeled.

De04 was used for the sheet material with an initial thickness of 0.75[mm] and a diameter of 160[mm]. A square deep-drawing tool corresponding to Benchmark tool for the NumiSheet’93 conference where applied and the produced cup was 38[mm] high and a blankholder force of 20 kN was applied. The elasto-plastic behavior was modeled, using exponential hardening and anisotropy behavior was modeled using Barlat’s 2D criterion.

1: Initialize the element thickness $e_{th}$, number of integration points $n_{ipt}$, material parameters and the Jacobian of the element and store the result in $V_i$.  
2: if $(t = 0.0$ and $j=1)$ then  
3: Calculate the volume of the $i$’th element and store the result in $V_i$.  
4: end if  
5: Update the stress and incremental plastic strain $d\sigma_{eq}$ for the $i$’th element and $j$’th integration point, see e.g. [20].  
6: if $(t > t_{count})$ then  
7: Write the Jacobian $J$ and incremental plastic work $dW$.  
8: $k = k + 1$  
9: $t_{count} = t_{count} + t_{dump}$  
10: end if  
11: Update the incremental plastic work array $dW_k = dW_{k} + V_{ipt} \sigma_{eq} d\sigma_{eq}$  
12: Update the Jacobian $J$  
13: $J_{k1} = J_{k1} + V_{ipt} \sigma_{eq} \frac{\partial \sigma_{eq}}{\partial x_{ipt}}$  
14: $J_{k2} = J_{k2} + V_{ipt} \sigma_{eq} \frac{\partial \sigma_{eq}}{\partial x_{ipt}}$  
15: $J_{k3} = J_{k3} + V_{ipt} \sigma_{eq} \frac{\partial \sigma_{eq}}{\partial x_{ipt}}$  
16: $J_{k4} = J_{k4} + V_{ipt} \sigma_{eq} \frac{\partial \sigma_{eq}}{\partial x_{ipt}}$  
17: $J_{k5} = J_{k5} + V_{ipt} \sigma_{eq} \frac{\partial \sigma_{eq}}{\partial x_{ipt}}$  
18: $J_{k6} = J_{k6} + V_{ipt} \sigma_{eq} \frac{\partial \sigma_{eq}}{\partial x_{ipt}}$  
19: End.
TABLE 4. Inverse identified parameters using a fixed value of the Barlat coefficient \(M=6.0\), and free friction coefficients, where equal friction coefficients are assumed between the tool parts and the blank.

<table>
<thead>
<tr>
<th>Initial set 2</th>
<th>Initial</th>
<th>Lower</th>
<th>Upper</th>
<th>Inverse</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu)</td>
<td>0.1</td>
<td>0.08</td>
<td>0.15</td>
<td>0.1186</td>
</tr>
<tr>
<td>(K)</td>
<td>550</td>
<td>500</td>
<td>650</td>
<td>593.14</td>
</tr>
<tr>
<td>(n)</td>
<td>0.24</td>
<td>0.18</td>
<td>0.30</td>
<td>0.2301</td>
</tr>
<tr>
<td>(a)</td>
<td>0.65772</td>
<td>0.6</td>
<td>0.8</td>
<td>0.7174</td>
</tr>
<tr>
<td>(h)</td>
<td>0.97621</td>
<td>0.95</td>
<td>1.1</td>
<td>1.0202</td>
</tr>
<tr>
<td>(p)</td>
<td>0.89716</td>
<td>0.85</td>
<td>1.00</td>
<td>0.9478</td>
</tr>
</tbody>
</table>

TABLE 5. Material parameters produced by uni-axial tensile tests.

<table>
<thead>
<tr>
<th>Uni-axial parameters</th>
<th>(K)</th>
<th>(n)</th>
<th>(R_{00})</th>
<th>(R_{45})</th>
<th>(R_{90})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>544.68</td>
<td>0.239</td>
<td>1.758</td>
<td>1.287</td>
<td>2.028</td>
</tr>
</tbody>
</table>

TABLE 6. Inverse identified parameters using a fixed value of the Barlat coefficient \(M=6.0\), and free friction coefficients, where equal friction coefficients are assumed between the tool parts and the blank.

<table>
<thead>
<tr>
<th>Lankford's coefficients</th>
<th>Initial 1</th>
<th>Inverse 1</th>
<th>Initial 2</th>
<th>Inverse 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu)</td>
<td>0.1</td>
<td>0.1163</td>
<td>0.1</td>
<td>0.1186</td>
</tr>
<tr>
<td>(K)</td>
<td>550</td>
<td>592.23</td>
<td>550</td>
<td>593.14</td>
</tr>
<tr>
<td>(n)</td>
<td>0.25</td>
<td>0.2309</td>
<td>0.24</td>
<td>0.2301</td>
</tr>
<tr>
<td>(R_{00})</td>
<td>1.8</td>
<td>1.9071</td>
<td>1.9</td>
<td>1.8926</td>
</tr>
<tr>
<td>(R_{45})</td>
<td>1.4</td>
<td>1.4045</td>
<td>1.4</td>
<td>1.3953</td>
</tr>
<tr>
<td>(R_{90})</td>
<td>2.3</td>
<td>1.7839</td>
<td>2.2</td>
<td>1.6923</td>
</tr>
</tbody>
</table>

FIGURE 2. Barlat’s 2D yield locus for a fixed \(\sigma_{12} = \frac{\sigma_{eq}}{10}\) [MPa] and an equivalent strain \(\varepsilon_{eq} = 0.5\). The two inverse identified yield surfaces are close to identical and a significant difference between the inverse and uni-axial parameters can be observed. Both the material parameters and friction coefficients are identified.

The largest error was located within the first 5 [mm] of the punch displacement (0.3 [kN]), this relative large error can be explained by the applied friction model utilizing Coulomb’s friction, i.e. a friction model using a dynamic and static friction coefficient will improve the correlation in this area. However, the error was reduced to \(\pm 0.1\) [kN] within the punch displacement range from 5 [mm] to 38 [mm].

The identified yield locus for both the initial parameter sets are close to identical, see figures 2 and 3. However, a fairly large difference can be observed between the inverse and uni-axial yield locus. The uni-axial tensile test appears to underestimate the strength coefficient \(K\) and to overestimate the hardening coefficient \(n\), and as a result the yield stress is underestimated, see tables 5 and 6.

FIGURE 3. Normalized yield locus \((\sigma_{12} = \frac{\sigma_{eq}}{10})\).

FIGURE 4. Illustration of fitted and empirical data for a square cup with a free friction coefficient.
An insignificant difference in the identified parameters was observed, the term insignificant is judged from close resembles between the identified yield loci, they are close to identical, see figures 2 and 3. Further, a very good fit between the experimental and fitted punch force was achieved, see figures 4 and 5, where the residuals were reduced to ±0.1[kN] over the majority of the punch stroke, whereas the largest error (0.3 [kN]) was observed within the first 5[mm] of the punch displacement, the relatively large error in this region may indicate that the Coulomb’s friction model based on a single friction coefficient is unsuitable in this region.

A difference between the uni-axial parameters and inverse parameters was observed, both with respect to anisotropy and hardening parameters. The uni-axial tension tests seems to underestimate the strength coefficient \( K \) and overestimate the hardening coefficient \( n \), and as a result the yield stress is underestimated, yielding a significant difference between the uni-axial and inverse identified yield loci.

Finally, a very simple strategy for analytically determination of friction sensitivity was proposed. The friction sensitivity scheme provided a good estimate of gradient if compared with the finite difference scheme and reduced the number of finite element analysis considerably compared with the finite difference approach.

**REFERENCES**