PERTURBATION OF THE COMPLIANCE FUNCTIONAL DUE TO THE APPEARANCE OF A SMALL CAVITY IN AN ELASTIC BODY

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Summary The paper proves that the known methods of assessing an increment of strain energy due to the appearance of small cavities in elastic solids lead to one equivalent result. The following approaches are discussed: the compound asymptotic method by Mazja, Nazarov and Plamenevskii, the topological derivative method by Sokolowski and Zochowski and the method of Eshelby. The result derived determines the characteristic function of the bubble method.

CHANGE OF ENERGY ACCORDING TO MAZJA-NAZAROV-PLAMIENIEVSKII AND ESHELBY

The present paper refers to the problem of assessing a change of a shape functional due to the appearance of a small cavity within the domain. The formulae expressing such changes play a crucial role in the evolutionary methods of topology optimization, see [1]. Finding the stiffest layouts needs assessing the change of the compliance or, equivalently, change of the elastic energy stored in the body. Let us start with the plane elasticity problem. Assume that a plane body is subjected to self-equilibrated tractions of intensity $\mathbf{p}$ on $\partial\Omega$. The elastic energy stored in the body equals

$$\mathcal{E}(\Omega) = \frac{1}{2}\int_{\Omega} \sigma^{\alpha\beta}(\mathbf{v})\epsilon_{\alpha\beta}(\mathbf{v})dx \tag{1}$$

where $\mathbf{v} = (v_1, v_2)$ represents the displacement field caused by $\mathbf{p}$. The stress field $\boldsymbol{\sigma}$ is linked with the deformations by $\boldsymbol{\sigma} = \boldsymbol{A}\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}(\mathbf{v})$ represents a symmetric part of $\nabla\mathbf{v}$. Let us consider now the domain $\Omega_\varepsilon = \Omega \setminus \overline{\omega}_\varepsilon$, where $\omega_\varepsilon = \{x \mid x/\varepsilon \in \omega\}$. We assume that $0 \in \omega$, ω being a rescaled domain of the opening; ε is a small parameter. The tractions $\mathbf{p}$ cause the displacement field $\mathbf{u}^\varepsilon$. The elastic energy

$$\mathcal{E}(\Omega_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} \sigma^{\alpha\beta}(\mathbf{u}^\varepsilon)\epsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)dx \tag{2}$$

should be greater than $\mathcal{E}(\Omega)$ irrespective of the shape of the cavity ω_ε. Note that the origin 0 lies within ω_ε for each $\varepsilon > 0$. We shall say that a hole ω_ε nucleates at 0. The case of $\varepsilon = 0$ refers to the case without any hole within Ω. Let us denote : $\boldsymbol{\epsilon}^0 = \boldsymbol{\epsilon}(\mathbf{v})(0)$ for the deformation state at point 0 caused by the loading $\mathbf{p}$.
The compound asymptotic method, exposed recently in Mazja et al.[4] leads to the following result

$$\mathcal{E}(\Omega_\varepsilon) = \mathcal{E}(\Omega) - \frac{1}{2}\varepsilon^n \varepsilon^0_{\alpha\beta} M^{\alpha\beta\lambda\mu}\varepsilon^0_{\lambda\mu} + o(\varepsilon^n) \ , \tag{3}$$

where $n = 2$ and $\boldsymbol{M} = \left(M^{\alpha\beta\lambda\mu}\right)$ will be called the Polya-Szegö tensor. The above result is reported without proof in Movchan and Movchan [5]; the details of the derivation are given in Lewinski and Sokolowski [3]. Tensor $\boldsymbol{M}$ characterizes the outer problem for ω

$$\left(P_\omega^{(\lambda\mu)}\right) \quad \left| \quad \begin{aligned} &\text{find } \boldsymbol{\chi}^{(\lambda\mu)} \text{ defined in } \mathbb{R}^2\backslash\overline{\omega} \text{ such that} \\ &A^{\alpha\beta\gamma\delta}\frac{\partial}{\partial y_\beta}\epsilon^y_{\gamma\delta}(\boldsymbol{\chi}^{(\lambda\mu)}) = 0 \quad \text{in } \mathbb{R}^2\backslash\overline{\omega}\ , && (4)\\ &A^{\alpha\beta\gamma\delta}\epsilon^y_{\gamma\delta}(\boldsymbol{\chi}^{(\lambda\mu)})\nu_\beta = -A^{\alpha\beta\lambda\mu}\nu_\beta \qquad \text{on } \partial\omega\ , && (5)\\ &\boldsymbol{\chi}^{(\lambda\mu)} \to \mathbf{0} \qquad \text{if } ||y|| \to \infty && (6) \end{aligned}\right.$$

Here $y = (y_1, y_2)$ parametrize the domain $\mathbb{R}^2\backslash\overline{\omega}$; $\epsilon^y_{\gamma\delta}(\mathbf{v})$ represents the symmetric part of $\nabla\mathbf{v}$ with respect to the Cartesian system (y_1, y_2); the Cartesian base vectors are denoted by $\boldsymbol{e}_1, \boldsymbol{e}_2$. The vector fields $\boldsymbol{\chi}^{(\lambda\mu)}$ are uniquely determined. The components of $\boldsymbol{M}$ are given by

$$M^{\kappa\delta\sigma\gamma} = -A^{\kappa\delta\sigma\gamma}|\omega| - \mathcal{M}^{\kappa\delta\sigma\gamma} \tag{7}$$

$$\mathcal{M}^{\kappa\delta\sigma\gamma} = A^{\alpha\beta\kappa\delta}\int_{\partial\omega} \chi^{(\sigma\gamma)}_\alpha \nu_\beta\, ds \ , \tag{8}$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2)$ is a unit vector outward normal to $\partial\omega$ and $|\omega|$ represents the area of ω. The tensor $\boldsymbol{M}$ is negative definite under very weak assumptions of regularity of ω, see Ref.[3].

The results (4)-(6) and (7)-(8) can be generalized to the $3D$ case. Then the exponent $n = 3$, see Eq.(3), the contour integral in (8) is replaced by the surface integral; $|\omega|$ stands for the volume of ω. Moreover, one can show that the known Eshelby formulae for the energy change are equivalent to the result .

CHANGE OF ENERGY FOUND BY THE TOPOLOGY DERIVATIVE METHOD

The topological derivative has been introduced by Sokolowski and Zochowski [9] in order to formulate necessary conditions of optimality for optimum shape design problems. The topological derivative measures a change of a functional caused by removing a small ball from the given domain. A method of computing the topology derivative relevant to weakening the domain by a small cavity (or hole) of arbitrary shape is proposed e.g., in Lewinski and Sokolowski [3]. The method is based on the velocity method of shape optimisation. The speed vector field is chosen as a linear function in the space variable. This method can be applied for confirming the formula (3) by a new manner. The final result has the form (3) with tensor $\boldsymbol{M}$ replaced by $\boldsymbol{G}$ given by

$$G^{\gamma\delta\iota\varrho} = -\frac{1}{2} A^{\alpha\beta\lambda\mu} \int\limits_{\partial\omega} \epsilon^{y}_{\alpha\beta}(\boldsymbol{\psi}^{(\iota\varrho)}) \epsilon^{y}_{\lambda\mu}(\boldsymbol{\psi}^{(\gamma\delta)}) \boldsymbol{y} \cdot \boldsymbol{\nu}\, ds \tag{9}$$

for the $2D$ case. Here

$$\boldsymbol{\psi}^{(\alpha\beta)}(y) = \boldsymbol{\chi}^{(\alpha\beta)}(y) + \boldsymbol{E}^{(\alpha\beta)}(y) \ , \tag{10}$$

where

$$\boldsymbol{E}^{(\lambda\mu)}(y) = \frac{1}{2}(y_{\lambda}\boldsymbol{e}_{\mu} + y_{\mu}\boldsymbol{e}_{\lambda}) \ , \qquad \lambda, \mu \in \{1, 2\} \tag{11}$$

Thus we have arrived at two seemingly different estimates of change of energy, since it is not easily seen why $\boldsymbol{G}$ should be equal to $\boldsymbol{M}$. It is, however, the case. The proof of the equality$\boldsymbol{G} = \boldsymbol{M}$ has been recently reported in Ref.[6].

APPLICATIONS

Topological derivatives are used in numerical methods of topology and shape optimization [1], [2]. Modelling of topological derivatives can be performed [6] by the selfadjoint extensions of elliptic operators with singular potentials [8]. The latter replace the geometrical singularities with the prescribed precision, confirmed by the error estimates. Moreover, new tools for the topology optimisation are provided by the exterior topology derivatives [7] relevant to the case of adding new material to a feasible domain. Some examples of applications are provided.

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