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COMPOSITE PLATES WITH ACTIVE FIBERS

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Summary The bending problem of composite plates reinforced with active fibers is considered. The fibers are made of a shape memory material which may undergo a martensitic phase transformation. The matrix is treated as an elastic medium. Due to the phase transformation in fibers the deformation process is dissipative and accompanied by hysteresis loops. A macroscopic model of the stress-induced phase transformation under isothermal conditions is applied. The dissipation mechanism is traced by imposing the requirements of the second principle of thermomechanics and making use of a set of internal variables describing distributions of austenite and different variants of martensite. This hysteretic problem is formulated in the form of a variational inequality of evolutional type. The finite dimensional counterpart of it is derived by the finite element method, and is solved incrementally as a sequence of complementarity problems. Results of numerical simulation will be presented.

INTRODUCTION

Novel, the so-called *smart*, *active* or *adaptive* materials allow one to construct new functional types of structures and devices; composite materials can be designed to effectively meet the highly specialized needs of advanced technology [3]. In this presentation we are concerned with the mathematical modeling of composite plates that are symmetrically reinforced with shape memory alloy (SMA) fibers. Perfect bonding between the fibers and the matrix is assumed. Depending on the stress level, the SMA fibers can undergo a martensitic phase transformation. The latter takes place at a micro scale and can be described as cooperative movements of atoms into a new more stable crystal structure. Martensitic transformation is the diffusionless, first-order transformation occurring by nucleation and growth. The parent phase is conventionally called *austenite*, whilst the product of the transformation, which may occur in many variants, is called *martensite*. Although, the martensitic phase transformation is accompanied by energy dissipation, it is crystallographically reversible as the thermoelastic shape memory alloys exhibit a unique ability to recover shape.

We consider a thin plate of total thickness h composed of N_l orthotropic layers, of which N_r are reinforced with SMA fibers. The total domain occupied by the plate is $\Omega_0 \times (-h/2, h/2)$, where Ω_0 stands for the undeformed midplane of the composite plate. Let $\boldsymbol{u} = (u, v, w)$ be the displacement vector with its components u, v, w along the Cartesian coordinate system axes $x\,y\,z$, respectively. Further, let $\boldsymbol{u}_0 = (u_0, v_0, w_0)$ denote the midplane displacements vector, in which $w_0 = w_0(x, y, t)$ is the deflection of the midplane and we set $u_0 = 0 = v_0$ neglecting the extension-bending coupling. In this evolutional nonlinear process t plays a role of a time-like parameter. In the classical Kirchhoff plate theory the nonzero strains written in vector form are $\boldsymbol{\varepsilon} \equiv \mathrm{col}\,(\varepsilon_{xx}, \varepsilon_{yy}, 2\varepsilon_{xy}) = -z\boldsymbol{\gamma}$, where $\boldsymbol{\gamma} = \mathrm{col}\,(\partial^2 w_0/\partial x^2, \partial^2 w_0/\partial y^2, 2\partial^2 w_0/\partial x \partial y)$. The kth layer is located between the points $z = z_k$ and $z = z_{k+1}$ in the thickness axis. We have supposed that the response of the matrix is linearly elastic, whilst that of fiber's material will be governed by the phase transformation rules in the range of isothermal pseudoelasticity.

The linear elastic constitutive equations $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$ between components of the stress $\boldsymbol{\sigma}$ and strain ε tensors can be written for the kth orthotropic layer in its principle material coordinates $x_1\,x_2\,x_3$ in the matrix form

$$\left\{\begin{array}{c}\sigma_1\\ \sigma_2\\ \sigma_6\end{array}\right\}^{(k)} = \left[\begin{array}{ccc}C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{66}\end{array}\right]^{(k)} \left[\begin{array}{c}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{array}\right]^{(k)}$$

where components C_{ij} of the elasticity matrix $\boldsymbol{C}$ define elastic properties of the layer. In the global coordinate system $x\,y\,z$, relations () become

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{array}\right\}^{(k)} = \left[\begin{array}{ccc}\bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16}\\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26}\\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66}\end{array}\right]^{(k)} \left[\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{array}\right]^{(k)} \tag{1}$$

where $\bar{\boldsymbol{C}}^{(k)} = \boldsymbol{T}^{(k)}\boldsymbol{C}^{(k)}\boldsymbol{T}^{(k)^{\mathrm{T}}}$, $\boldsymbol{T}^{(k)}$ – the transformation matrix from $x_1\,x_2\,x_3$ to $x\,y\,z$ for the kth layer, see e.g. [10].

CONSTITUTIVE RELATIONS FOR A SMA LAYER

The shape memory alloy is a multiphase material that may occur in $N+1$ preferred strain states $\boldsymbol{d}_i$ which correspond to austenite and N variants of martensite, whose volume fractions c_i are changing during the martensitic phase transformation process. At fixed volume fractions $\boldsymbol{c}$, the averaged Helmholtz free energy of the austenite-martensite mixture may be expressed as a locally quadratic multi-well function, cf e.g. [1, 4, 9],

$$W(\varepsilon, \boldsymbol{c}) = \frac{1}{2}\left[\varepsilon - \boldsymbol{d}(\boldsymbol{c})\right]\cdot\boldsymbol{C}\left[\varepsilon - \boldsymbol{d}(\boldsymbol{c})\right] + \sum_{i=1}^{N+1} c_i \varpi_i + \frac{1}{2}\sum_{i=1}^{N+1}\sum_{j=1}^{N+1} c_i c_j B_{ij}$$

wherein $\boldsymbol{d}(\boldsymbol{c}) = \sum_{i=1}^{N+1} c_i \boldsymbol{d}_i$ is the effective transformation strain, $\boldsymbol{C}$ the (same) elasticity tensor for each phase, and ϖ_i,B_{ij} material constants. From the 2nd principle of thermomechanics we can obtain an expression for the driving force of phase transformation which must satisfy the positive dissipation constraint [6]

$$\mathcal{D} = -\frac{\partial W}{\partial \boldsymbol{c}}\cdot\dot{\boldsymbol{c}} \equiv \boldsymbol{X}\cdot\dot{\boldsymbol{c}} = \sum_{m=1}^{N} X_m \, \dot{c}_m \geq 0 \tag{2}$$

Accounting for (2) we introduce the threshold functions κ_m for the driving force X_m $(m = 1, \ldots, N)$ and define the phase transformation functions $G_m = G_m(\varepsilon, \boldsymbol{c})$ whose positive $()^+$ and negative $()^-$ parts are

$$G_m^+ \equiv \kappa_m^+ - X_m \geq 0, \qquad G_m^- \equiv X_m - \kappa_m^- \geq 0$$

The evolution of volume fractions c_m of martensite variants are governed by the following conditions

$$c_m \geq 0, \quad \sum_{m=1}^{N} c_m \leq 1, \quad \sum_{m=1}^{N} G_m^{\pm}(\boldsymbol{c})(\xi_m^{\pm} - \dot{c}_m^{\pm}) \geq 0 \quad \text{for all } \xi_m^{\pm} \geq 0 \tag{3}$$

The stresses in the reinforced rth layer are, cf (1),

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{array}\right\}^{(r)} = \left[\begin{array}{ccc}\bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16}\\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26}\\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66}\end{array}\right]^{(r)} \left[\begin{array}{c}\varepsilon_{xx} - d_{xx}(\boldsymbol{c})\\ \varepsilon_{yy} - d_{yy}(\boldsymbol{c})\\ \varepsilon_{xy} - d_{xy}(\boldsymbol{c})\end{array}\right]^{(r)} \tag{4}$$

CLOSING REMARKS

Using (1) and (4) we arrive at the differential equation for the plate,

$$L\left(\boldsymbol{D}\,\boldsymbol{\gamma}(w_0)\right) + L\left(\boldsymbol{S}\,\bar{\boldsymbol{d}}(\boldsymbol{c})\right) = p(x, y, t) \tag{5}$$

wherein $L(\cdot)$ is the linear partial differential operator of an anisotropic plate, p is loading, and

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N_l} \bar{C}_{ij}^{(k)}\left(z_{k+1}^3 - z_k^3\right), \qquad S_{ij} = \frac{1}{2}\sum_{r=1}^{N_r} \bar{C}_{ij}^{(r)}\left(z_{r+1}^2 - z_r^2\right)$$

The equlibrium condition (5) and the phase transformation conditions (3) will be written in the form of a variational equation and variational inequalities, respectively, from which we can determine the deflection w_0 and distributions $\boldsymbol{c}^{(r)} = \boldsymbol{c}^{(r)}(x, y, t)$ of different variants of martensite in all reinforced layers of the plate. A computer code for this evolutional boundary value problem has been developed, based on the standard FEM [5]. Results of numerical examples will be presented. As the next step, it would be interesting to apply hierarchical models [7, 8], and to consider the case of magnetic shape memory alloy fibers where an adaptive procedure could be used [2].

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