TOPICS IN ASTROPHYSICAL FLUID DYNAMICS

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Abstract This brief description of some fluid dynamical problems of astrophysical interest focuses on two effects that are characteristic of the subject: self-gravity and radiative forces. Self-gravity is important in determining the basic structures of cosmic bodies as well as producing some intriguing instabilities. Radiation, by which we observe these bodies, produces forces that may be disruptive to their basic structures and be a source of vigorous fluid dynamical activity in the form of photon bubbles and radiatively driven vortices.

Keywords: Gravitational instability, solitary waves, astrophysical vortices, photon bubbles

This sampler of problems in AFD begins with the classical problem of gravitational instability and continues with some current problems that I find very intriguing. Given the imposed space limitations, I can only touch on these and I need to be very stingy with references (not of my own, of course) and will assume that I need not provide many fluid references for the expected readership of this volume. (But feel free to use the email address above.)

1. Gravitational Instability

Massaged Models

The observable part of our universe looks homogeneous in the large, but it is clearly rather lumpy on smaller scales. There are stars, clusters of stars, galaxies and clusters of galaxies. Possibly there are even larger structures but here we come to a less clearly defined topic where people may yet argue about how (or whether) to study such issues as the dimension of the point set that approximates the distribution of galaxies. But it is undoubtedly interesting to inquire into the origin of the inhomogeneities of the universe. Our current understanding of cosmology is that the early universe was dense and hot and very dissipative. Hence it is thought to have been rather homogeneous in the earliest times that we can reasonably think about and the lumpiness is considered to have been caused by gravity.

The breakup of a homogenous fluid into reasonably discrete structures is believed to be caused by an instability whose qualitative nature was already imagined by Newton. Jeans [13] was apparently the first to formulate this problem. He assumed a perfect barotropic fluid with self-gravity and used these simple equations of motion:

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) = -\nabla p - \rho\nabla V, \tag{1}$$

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{2}$$

$$\nabla^2 V = 4\pi G\rho, \tag{3}$$

$$p = K\rho^\Gamma. \tag{4}$$

Jeans posited a static homogenous solution to these equations (though one presumes that he knew better) and studied perturbations on this "solution". But there is no infinite, homogenous, static, self-gravitating medium in cosmology, not even in Einstein's original theory of gravity [11]. Einstein added a term to his equation for the curvature of spacetime that allowed him to find a static solution, unstable though it is. Similarly, we may add a term to the right side of the Euler equation to provide our Newtonian universe with a static solution. This term, $\rho\lambda\mathbf{r}$, represents a repulsive force, as Eddington [10] pointed out. Now we may introduce $\tilde{V} = V - \frac{1}{2}\lambda r^2$ and replace (1) and (3) by

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) = -\nabla p - \rho\nabla\tilde{V}, \tag{5}$$

$$\nabla^2\tilde{V} = 4\pi G(\rho - \rho_\lambda) \tag{6}$$

where $\rho_\lambda = \lambda/G$. This recalls the device used in plasma physics to maintain charge neutrality and we may now find a homogenous static solution and study its stability to clumping.

Einstein's extra term, "the cosmological term", intrigued people from its beginning, but its status was uncertain until recently. The recent discovery that the expansion of the universe is accelerating has brought this extra term into vogue, though its physical meaning remains unclear. Does the cosmological term mean what the simple statement given here suggests — the existence of matter with negative gravitational mass [21]? If so, we should perhaps add a fluid equation for this odd material as in the two-fluid plasma model. Many cosmologists look on the added term as a negative pressure that can drive the expansion velocity ever upward. Here it merely serves Einstein's original purpose for it: to produce a static solution.

The Dispersion Relation

To study the stability of the homogenous medium, we let $\rho = \rho_0 + \delta\rho$, $\tilde{V} = \tilde{V}_0 + \delta\tilde{V}$ and so on. With the extra term in place, we may assume that the basic fields are constant. If we then linearize the equations about the static state, we are led by standard manipulations to a Klein-Gordon equation for $\delta\rho$:

$$\left(\partial_t^2 - \nabla^2\right)\delta\rho = (4\pi G\rho_0)\delta\rho. \tag{7}$$

For plane waves with $\delta\rho \propto \exp(i\omega t - i\mathbf{k}\cdot\mathbf{x})$, we obtain the dispersion relation

$$\omega^2 = c^2(k^2 - k_J^2) \quad \text{with} \quad k_J^2 = (4\pi G\rho_0)/c^2, \tag{8}$$

and $c^2 = \Gamma p_0/\rho_0$.

The quantity k_J is called the Jeans wavenumber and, when the wavenumber of the perturbation is much larger than this, we recover ordinary sound waves. But for $k < k_J$, ω^2 becomes negative and we have instability. Because of the difference in sign from the electrostatic case, we get instability rather than oscillation. The Jeans length $1/k_J$ tells us the scale on which the gravity just balances the pressure gradient much as the balance of surface tension and pressure dictates the size of a liquid drop. When the length scale of a perturbation is larger than this critical size, collapse occurs. For a perfect gas with temperature T, we find from these formulae that $k_J^{-1} \approx 100\sqrt{(T/n)}$ light years where n is the number density and the collapse time for very small k is $\approx 5 \times 10^7/\sqrt{n}\,$yr.

Cosmology's Fictitious Forces

To study the formation of inhomogeneities in the universe properly, we need to take account of its expansion. This was first done for the linear stability problem by Lifshitz in 1946 in the context of relativistic cosmology, but the Newtonian case conveys the idea [5]. Even that story is on the long side so, to indicate how the expansion reduces the degree of instability, a brief look at the kinematics in an expanding medium may suffice.

For a universe that is inifinite and homogeneous in the large, it does not matter where we put the origin of coordinates, so let us presume that one has been chosen. The position with respect to the arbitrary origin of a fluid element is $\mathbf{r}$. We then transform to isotropically expanding coordinates such that

$$\mathbf{r} = R(t)\mathbf{x} \tag{9}$$

where $R(t)$ is a nondimensional function that tells us how the global scale of the universe changes in time. For some suitable origin of time we let t_0 be the present and take $R(t_0) = 1$. We find

$$\dot{\mathbf{r}} = \dot{R}\mathbf{x} + R\dot{\mathbf{x}} = H\mathbf{r} + R\dot{\mathbf{x}} \tag{10}$$

where $\mathbf{v} = H\mathbf{r}$ is the *Hubble flow* or global expansion and $H_0 = H(t_0)$ is called the Hubble constant. The acceleration of the fluid element is then

$$\ddot{\mathbf{r}} = R\left(\ddot{\mathbf{x}} + 2H\mathbf{x} + \dot{H}\mathbf{x} + H^2\mathbf{x}\right). \tag{11}$$

In the expanding coordinates, we acquire three additional terms or fictitious forces reminding us of those gained in going into a rotating frame. The scalar H plays the role of the rotation rate in that comparison, hence the analogue of the Coriolis force is a drag term. Material particles moving through an expanding medium are slowed down with respect to the background. This is an effect analogous to the cosmological redshift of photons. (You may think of this as a stretching of the de Broglie wavelengths.) Since H is not a constant, we get a force $\dot{H}\mathbf{x}$ corresponding to the Euler force. Finally there is the analogue of the centrifugal force, $H^2\mathbf{x}$. These extra terms appear in the Euler equation when we go to expanding coordinates.

The main point is that the cosmological drag term inhibits the development of gravitational instability but it does not kill it completely in standard cosmological models. Rather, it converts the exponential growth to an algebraic growth. It appears that this feeble instability may suffice to produce the structures we see in the universe around us according to many simulations. Still, if you want to get your hands analytically on the way gravitational instability develops, as one may do for weak instabilities, it is best to consider another static configuration of the mass.

Polytropic Slabs

If, in equation (4), you treat Γ as a parameter and not necessarily the ratio of specific heats, you have what is called the polytropic gas law. The spherically symmetric, static, self-gravitating solutions of Eqs. (1)–(4), served as models of stars in the nineteenth century and they remain qualitatively useful even now. The disks in spiral galaxies may also be modeled as polytropes to good effect and these are useful for studying their gravitational instability.

As a prelude to studying fluid dynamics in disks, one may study simpler flattened objects such as polytropic layers or slabs. The simplest case is a static configuration with the layer extending infinitely in two directions that we may call horizontal. The third, or vertical, direction is given the designation z. Just as for a stratified atmosphere, we may write the hydrostatic condition, the only difference being that the gravity is not specified but is governed by the Poisson equation.

Static solutions have $\rho = \rho_0(z)$ with

$$\rho_0(z) = \rho_0(0)\mathcal{C}_\Gamma(z) \tag{12}$$

where $\mathcal{C}_\Gamma(z)$ is given by a simple integral that, for general Γ, is a beta function. Several special cases may identified, notably

$$C_1 = \operatorname{sech}^2(k_J z), \quad C_2 = \cos(k_J z). \tag{13}$$

Here

$$k_J^2 = \left[\frac{4\pi G\rho_0}{c^2}\right]_{z=0} = \frac{4\pi G}{\Gamma K}[\rho_0(0)]^{2-\Gamma}. \tag{14}$$

The characteristic thickness of the slab is $\sim 1/k_J$ as before but now we have a static solution with no artifice. The model is admittedly simplified but it has scope for interesting dynamics. For instance, the distribution of density on the midplane controls the layer thickness and this may vary in the rotating case so that we may find Rossby waves propagating through disks.

Ledoux [14] derived the marginal stability condition for linearized perturbations on the isothermal slab ($\Gamma = 1$). He found two horizontal wavelengths that are marginal, with $k_{\rm hor} = 2k_J$ and 0. Here is a situation in which we can make use of asymptotic approaches for the modes of long wavelength where the instability first arises weakly. This is unlike the homogenous case, for which the maximum growth rate occurs at $k = 0$. The real case is even more favorable to this approach since disks of galaxies are embedded in very massive halos that make the disk thicknesses even less than the Jeans lengths of the disks. The halos themselves, though not visible, are generally believed to exist on the basis of their gravitational effects. Their influence strengthens the validity of the the thin-layer approximation.

Linear theory reveals that there are acoustic modes and gravity modes, just as in models of standard atmospheres. The surprise is that the instability occurs in the gravity modes and not in the acoustic modes as in the original Jeans problem. The gravity waves combine both thickness variations and true density variations, so have all that is needed. Since the largest scales are nearly marginal, one can assume slow times and long lengths to develop shallow layer theories [18]. With the additional restriction to small amplitudes, as in weakly nonlinear theories, one may derive nonlinear wave equations in the manners of Boussinesq and Korteweg and de Vries [19].

The familiar form of the theory is modified by a term representing the effect of self-gravity that comes in through the Poisson equation. For instance, in the case of nonlinear waves of small amplitude in a thin layer with $\Gamma = 2$ (the easiest case) one finds for the surface deformation that:

$$\eta_T - 3\eta\eta_X + \frac{1}{2}\eta_{XXX} = \mu\mathcal{H}[\eta] \tag{15}$$

where the Hilbert transform is

$$\mathcal{H}[\eta] = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\eta(Y)}{Y-X}dY \tag{16}$$

and X and T are suitably stretched variables. This equation has pole solutions, but we cannot say whether it is completely integrable. However, there are certainly solutions resembling solitons. The interest in such a result is that it has sometimes been thought that the highly dispersive nature of the waves in this kind of problem would not allow the formation of coherent structures. In fact, the nonlinearity in this problem leads to the formation of long-lived nonlinear waves.

What happens in the more realistic case of a rotating disk? This is an issue that is unresolved as yet in the nonlinear case. The finiteness of the disk brings discrete modes into play and they behave chaotically [4]. Still, some global order may emerge, perhaps with the help of other effects.

2. Astrophysical Vortices

The masses of stars range from 60-80 times the mass of the sun (4×10^{33} gm) down to a few tenths of a percent of the solar mass, that is, down to the masses of the giant planets. The massive stars have very hot atmospheres, (tens of thousands of Kelvins), while the low mass ones have much cooler atmospheres as a rule (6000 K for the sun). At both ends of this spectrum, atmospheric turbulence is observed. At the cool end, this is caused by thermal convection resulting from the lowering of the thermal conductivity through the raising of opacity by partially ionized hydrogen. The ionization of hydrogen also favors convection by raising the specific heat. In the atmospheres of the hottest stars, turbulence is detected through broadening of the spectral lines and it is often supersonic. The origin of this turbulence is not agreed upon, though there is no shortage of possible sources. Hot atmospheres are fully ionized and are not subject to thermal convection in the usual way, but they are rapid rotators and they pulsate. When the pulsation is vigorous, thermal convection may be driven parametrically. Moreover, hot stars have high radiation pressure that provokes instability and complicates the dynamics. Already at the qualitative level there are some interesting fluid dynamical issues.

We see spots on the sun and these are caused by magnetic flux tubes that protrude from the solar surface. The fields are locally strong enough to inhibit the convective transport of heat outward and so relatively cool (but still quite warm) spots are produced. By contrast, in Jupiter's atmosphere, a much cooler place, we find evidence for vortex tubes at the surface. Since there is a full range of masses between the two limits (sun and Jupiter) we may someday be able to observe the transition between the two kinds of coherent structures, but there is no reason why this transition could not be studied theoretically at present, perhaps numerically. This is a potentially revealing instance of the transition between the purely fluid and the magnetofluid regimes.

Related questions arise in the study of the accretion disks that form around condensed objects on many scales. These are rather different from the disks in spiral galaxies. Accretion disks represent inflows of mass from various sources such as companion stars in the case of binary stars, to the ambient stars around massive black holes in the centers of galaxies. The primitive nebula that might have preceded the formation of our solar system, as Kant and Laplace first suggested, are cooler examples of this kind of structure. As the matter flows toward the central object, its net angular momentum makes itself felt and a disk is formed. Before it can settle into the central object, the inflowing matter must get rid of its angular momentum. Various mechanisms have been proposed for expelling the angular momentum, mostly calling on some form of turbulence though waves and magnetic fields have been considered. Vortices could also play a part in the process. There are some similarities here to the flow around the polar vortex on earth that is central to the ozone problem, though accretion disks frequently are magnetized.

The large scale flow in accretion disks is governed mainly by Kepler's laws so that the circular velocity around the central object in an axially symmetric disk varies like $1/\sqrt{r}$. This represents a linearly stable shear and it is not yet decided whether nonlinear instability can occur in such a flow. (I first heard people arguing about that some thirty-five years ago.) However, as Chandrasekhar and others showed, magnetic fields can catalyze the conversion of the energy in the rotational flow of disks into turbulence, but it was some time till the importance of this result for disks was appreciated [3]. Can the resulting disorder lead to the formation of vortices?

Twenty-five years ago I put a drawing of a disk with whorls in it in an article on turbulence for an encyclopaedia aimed at twelve-year olds. The hope was that in ten years one of them would appear in my office with a fully completed simulation that revealed what the disk really looked like. This did not happen, so the next message in a bottle was sent as a remark at the end of a paper on vortices in stars and planets [9]. This got a response from P.A. Yecko whose thesis revealed that large-scale spiral vortices formed. (At the suggestion of A. Ingersoll, Yecko adapted a code written by E. Chassignet for simulating the oceanic thermocline.) Several astronomers objected that the Keplerian shear in disks would shred a vortex. The shredding is avoided by anticyclonic vortices which shield themselves with protective cocoons of reduced shear. Several subsequent simulations with higher resolution have shown their robustness [6, 12, 15]. A recent, two-d, compressible simulation at high resolution by G. Murante and colleagues in Torino strikingly shows how a single anticyclonic vortex survives in the Keplerian shear, at least for the ten disk rotations they followed. That vortex also generated large-scale spiral extensions in line with Yecko's results. The issue of what magnetic fields do to these processes needs clarification as do theoretical questions about how vortices form. But their existence would play a role in forming observable inhomogeneities on disks [1]. An example of a disk simulation in a two-dimensional Keplerian flow with slowly decaying turbulence is shown after [7] in Fig. 1.

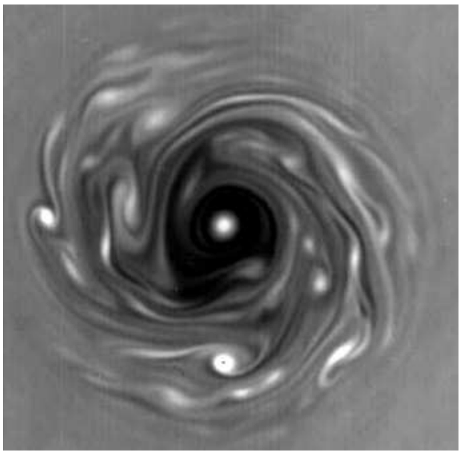

Figure 1. Vortices on a Keplerian Disk

3. Photofluiddynamics

The radiation that permits us to observe cosmic bodies also plays a role in their formation, structure and evolution. The thermal aspects of the radiation are familiar to fluid dynamicists, at least qualitatively. What may be of more interest in an introduction to AFD is the way that the force of radiation on matter may influence the dynamics of the ambient material medium. The phenomena arising in this subject seem sufficiently different from ordinary fluid dynamics that I have followed the advice of M. E. McIntyre and sought a suitably distinctive terminology. The name used here for the subject is inspired by Lighthill's "Biofluiddynamics." What follows is a very brief introduction to some aspects of this subject. For some background on the equations see [16].

The Radiative Fluid

The most direct derivation of the basic equations of fluid dynamics subject to stresses from a coexisting radiation field is by way of the transport equations for radiation and matter. These are kinetic equations that contain terms representing the interaction of the matter with the radiation. This somewhat technical problem is best formulated by treating the radiation fluid as a gas of photons. Nevertheless, there are some real complications depend on the frequencies of the photons and the state of the matter. Such details are out of place here, so we simply assume that the matter is grey, that is, indifferent to the frequencies of the photons passing through it. We shall also omit the details of the state of the matter such as the degree of ionization, which we shall take to be complete for very hot objects. So we may go straight to the first two moments of the transfer equation which, for a gas of photons, are equations for the moments of the distribution function or specific intensity.

Let $\mathcal{E}$, $\boldsymbol{\mathcal{F}}$ and $\mathbb{P}$ be the energy density, energy flux and pressure tensor of the radiation fluid. These satisfy the two moment equations

$$\partial_t \mathcal{E} + \nabla \cdot \boldsymbol{\mathcal{F}} = \text{interactions with matter} \tag{17}$$

and

$$\partial_t \boldsymbol{\mathcal{F}} + c^2 \nabla \cdot \mathbb{P} = -\rho\kappa c\, \boldsymbol{\mathcal{F}} \tag{18}$$

where $(\rho\kappa)^{-1}$ is the mean free path of a photon. We see that the latter equation may be written as

$$\nabla \cdot \mathbb{P} = -\frac{\rho\kappa}{c} \boldsymbol{\mathcal{F}} + \mathcal{O}(c^{-2}). \tag{19}$$

Naturally, we face the usual problem of closing off the moment hierarchy and here we take the simplest closure, $\mathbb{P} = \frac{1}{3}\mathcal{E}\mathbb{I}$. Thus we ignore the (sometimes important) radiative contribution to the viscous stress. In some limits, we may argue from (17) that the divergence of the flux is quite small. (In effect, we are leaving out retardation terms with this approximation.) This is about as simple as we can make this problem, yet much of interest remains as we see when we write the equation for the material fluid as

$$\rho(\partial\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p - \frac{1}{3}\nabla\mathcal{E} - \rho\nabla V. \tag{20}$$

To this we add the mass conservation equation and the radiative equations which have been reduced to the diagnostic conditions

$$\nabla\cdot\boldsymbol{\mathcal{F}} = \mathbf{0} \quad \text{and} \quad \nabla\mathcal{E} = \frac{1}{3}\rho\kappa c\boldsymbol{\mathcal{F}}. \tag{21}$$

Radiatively Induced Instabilities

To see in what kind of conditions radiative effects may become important dynamically, consider the simple case of a plane-parallel medium stratified under gravity. The foregoing equations show that the hydrostatic condition is

$$\frac{1}{\rho}\frac{dp}{dz} = -\frac{1}{3}\nabla\mathcal{E} - \rho\nabla V = \frac{1}{3}\kappa c\boldsymbol{\mathcal{F}} - \rho\nabla V \tag{22}$$

where z is the upward vertical coordinate. In the typical case where the flow of radiation is outward, hence upward, the radiative force is one of levitation and it balances the gravitational attraction downward when the right side of this equation vanishes. That condition is called the Eddington limit. Near this condition, we find instabilities of both sound waves and gravity waves [22]. These are induced by both the thermal and dynamical effects of radiation and there is still room for a better physical understanding of these processes.

Those familiar with fluidized beds will recognize a commonality between the Eddington limit and the onset of fluidization, although the medium being traversed by the radiation is a fluid even below the critical condition. As in fluidization, the material layers are rendered unstable by the traversing fluid, though the details of the instability mechanisms do differ. As in those cases where voids form in fluidized beds, the radiative fluid is much less dense than the particles of the medium. This and other arguments suggest that photon bubbles will form in hot stars near to the Eddington limit [17]. Approximate solutions for photon bubbles can be constructed in the way that this is done in the theory of fluidization [20]. Related discussions have been given in the context of convection in the chimerical supermassive hot objects [23] and polar caps of pulsars where the magnetic fields are strong (10^{12} gauss) [2].

An interesting aspect of this process bears on the question of the lifetimes of objects that exceed the Eddington limit. In the case of fluidized beds, flow through the bed in excess of the value needed for fluidization is observed to escape in voids, or bubbles. We may similarly expect cosmic bodies to survive above the Eddington limit. A modified limit needs to be calculated but another feature of this problem needs to be addressed first.

Photovorticity

The hottest objects are relatively rare and there are very few nearby. We see them because they are intrinsically luminous but it is hard to know whether they are like other rotating, turbulent cosmic bodies in forming vortices or concentrated magnetic flux tubes. However, it is clear that the hot objects do rotate rapidly and are turbulent in their outer layers. Moreover, there are observational grounds for supposing that there are spots on hot stars [8] and disks [1]. What can cause spots in such conditions?

Hot stratified media are unstable and, as for fluidized beds, we may expect the formation of (photon) bubbles in objects near the Eddington limit. If we make a vortex in such conditions, we anticipate that, as in many laboratory experiments on rotating turbulence, bubbles are attracted into vortices. Indeed, it is the practice to use small bubbles as markers of vortices in such experiments. On the other hand, bubbles flowing into a vortex will bring with them angular momentum and, when this is of the right sign, the vortex will be intensified. We have in such a situation the makings of an instability for vortex production analogous to what has been seen in laboratory experiments [24].

Vortex formation in hot media is of interest since it would lead to strong inhomogeneities in the emerging radiation field that would have diagnostic implications. It would also be important in the fluid dynamics of hot objects since a vortex of the right kind is a conduit through which radiation may escape from a hot object or disk without disrupting it. A simple calculation reveals the nature of this process.

Let us omit the complications of global rotation and consider an isolated vortex in a stratified polytropic fluid with radiation coming from below [9]. A standard vortex with gravity balancing the pressure gradient in the vertical direction and the centrifugal force in the (horizontal) radial direction is readily constructed. Then, in radiative equilibrium, we have $\mathcal{E} = aT^4$, where T is the temperature. For radiative prob-

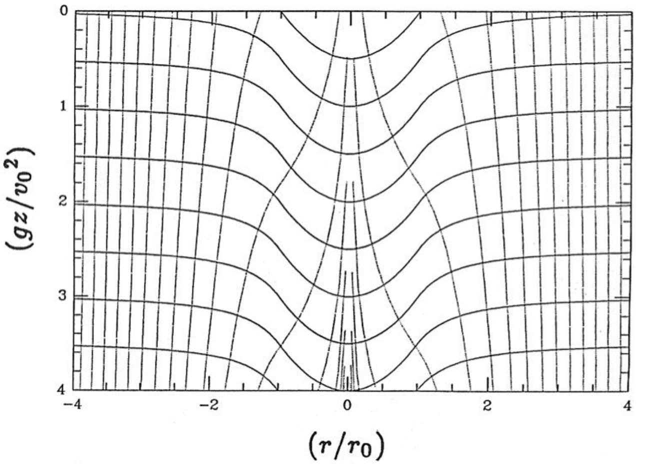

Figure 2. Flow through a phortex

lems the parameter value $\Gamma = 4/3$ is frequently adopted. This value has the advantage of making ρ^3/T a constant and that greatly simplifies the second of (21). That pair of equations is then readily solved and the streamlines of the radiative energy flux $\mathcal{F}$ are as shown in Fig. 2; the contours of T are also indicated. This vortex provides not only a safety valve by which the radiation may escape but such a beam should make itself apparent in observations of hot stars and disks.

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