EXTREME VALUE DISTRIBUTION AND RELIABILITY OF STOCHASTIC STRUCTURES

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Summary An original method to compute the extreme value distribution and dynamic reliability of stochastic structures is presented. A virtual stochastic process, related to the extreme value of the dynamic responses of stochastic structures, is constructed firstly, such that the extreme value is the sectioned random variable. A joint probability density equation is then deduced with the probability density evolution method (PDEM), which has been developed in the recent years and can give the instantaneous probability density function (PDF) of the dynamic responses, and numerically solved to give the PDF of the extreme value. The dynamic reliability is then easily computed by a simple integration over the safe domain. The comparison with the Monte Carlo simulation shows that the proposed method is of accuracy and efficiency.

INTRODUCTION

In the past 3 decades, many endeavours have been devoted to analysis of stochastic structures involving random structural parameters. The developed approaches, including the statistical and non-statistical methods, say, the random perturbation method and the orthogonal polynomial expansion method, are mostly aiming at probability characteristics such as second order statistics. On the other hand, in dynamic reliability assessment of stochastic structures, the level-crossing process theory, based on the Rice formula and the assumptions of out-crossing events, is widely used. Nonetheless, the joint PDF needed in the Rice formula is usually unavailable by the aforementioned methods dealing with stochastic structures. Moreover, the assumptions of out-crossing events will lead to additional errors.

As is well known that the extreme value distribution (EVD) is of paramount importance [2]. However, there have been so far few, if any, reports on the EVD of the dynamic responses of stochastic structures. In the present paper, a virtual stochastic process is constructed related to the extreme value of the responses of stochastic structures. With the recent developed probability density evolution method [3, 4], a joint probability density evolution equation is deduced and solved to give the EVD. The dynamic reliability is then easily obtained by the integration over the safe domain.

THE PROBABILITY DENSITY EVOLUTION METHOD FOR DYNAMIC RESPONSE ANALYSIS

For a general multi-degree-of-freedom system

$$\boldsymbol{M}(\boldsymbol{\Theta})\ddot{\boldsymbol{X}}+\boldsymbol{C}(\boldsymbol{\Theta})\dot{\boldsymbol{X}}+\boldsymbol{K}(\boldsymbol{\Theta})\boldsymbol{X}=\boldsymbol{F}(\boldsymbol{\Theta},t) \tag{1}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ are the n by n mass, damping and stiffness matrices, respectively, $\boldsymbol{F}$ is the n by 1 excitation vector, $\boldsymbol{\Theta}$ is the $n_{\boldsymbol{\Theta}}$ by 1 random parameters vector, involved in the structure and excitation, with the joint PDF $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$.

If the structural dynamics problem in (1) is well posed, the responses, under the deterministic initial condition $\dot{\boldsymbol{X}}(0)=\dot{\boldsymbol{x}}_0$, $\boldsymbol{X}(0)=\boldsymbol{x}_0$, of the stochastic structure are existent, unique and dependent on $\boldsymbol{\Theta}$, namely, $\boldsymbol{X}=\boldsymbol{G}(\boldsymbol{\Theta},t)$. Its component form is $X_l=G_l(\boldsymbol{\Theta},t)$, where X_l is the l-th component of $\boldsymbol{X}$, G_l is the l-th component of $\boldsymbol{G}$.

Denote the joint PDF of $(X_l,\boldsymbol{\Theta})$ as $p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)$, then there is $p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)=p_{X_l|\boldsymbol{\Theta}}(x,t\,|\,\boldsymbol{\theta})p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})=\delta(x-G_l(\boldsymbol{\theta},t))p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$, where $p_{X_l|\boldsymbol{\Theta}}(x,t\,|\,\boldsymbol{\theta})=\delta(x-G_l(\boldsymbol{\theta},t))$, is the conditional PDF, $\delta(\cdot)$ is the Dirac's function. Differentiate Eq.(2) in terms of t on both sides and rearrange it will lead to [3]

$$\partial p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)\big/\partial t+H_l(\boldsymbol{\theta},t)\cdot\partial p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)\big/\partial x=0 \tag{2}$$

where $H_l(\boldsymbol{\theta},t)=\partial G_l(\boldsymbol{\theta},t)\big/\partial t=\dot{X}_l$. The initial condition for Eq.(2) reads $p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)\,|_{t=0}=p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\delta(x-x_{l,0})$, where $x_{l,0}$ is the l-th component of $\boldsymbol{x}_0$.

Eq.(2) can be numerically solved combining the deterministic time-integration of Eq.(1) and the finite difference method with a TVD schemes, for the detail, see [4]. Then the PDF of $X_l(t)$, denoted as $p_{X_l}(x,t)$, can be obtained by $p_{X_l}(x,t)=\int_{\Omega_{\boldsymbol{\Theta}}}p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)\mathrm{d}\boldsymbol{\theta}$, where $\Omega_{\boldsymbol{\Theta}}$ is the distribution domain of $\boldsymbol{\Theta}$.

THE EXTREME VALUE DISTRIBUTION OF STOCHASTIC STRUCTURAL RESPONSES

With the analogous idea, the EVD of the stochastic structural responses can be obtained. For the well-posed problem (1), the extreme value of the response is existent, unique and dependent on $\boldsymbol{\Theta}$, i.e.,

$$Z_l=\sup_{\tau\in[0,T]}X_l(\tau)=W_l(\boldsymbol{\Theta},T) \tag{3}$$

Construct a virtual stochastic process $Q_l(\tau)$, $Q_l(\tau)=Z_l\cdot\tau=W_l(\boldsymbol{\Theta},T)\cdot\tau$.

It is obvious that Z_l is the sectioned random variable of $Q_l(\tau)$, i.e., $Z_l = Q_l(\tau)|_{\tau=1}$.
Similar to Eq.(2), denote the joint PDF of $(Q_l, \boldsymbol{\Theta})$ as $p_{Q_l\boldsymbol{\Theta}}(q,\boldsymbol{\theta},\tau)$, we have

$$\partial p_{Q_l\boldsymbol{\Theta}}(q,\boldsymbol{\theta},\tau)\big/\partial\tau + W_l(\boldsymbol{\theta},T)\cdot\partial p_{Q_l\boldsymbol{\Theta}}(q,\boldsymbol{\theta},\tau)\big/\partial q = 0 \tag{4}$$

with the initial condition $p_{Q_l\boldsymbol{\Theta}}(q,\boldsymbol{\theta},\tau)|_{\tau=0} = \delta(q)p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. Numerically solving Eq.(4) and integration will give the PDF of $Q_l(\tau)$, i.e., $p_{Q_l}(q,\tau) = \int_{\Omega_{\boldsymbol{\Theta}}} p_{Q_l\boldsymbol{\Theta}}(q,\boldsymbol{\theta},\tau)\mathrm{d}\boldsymbol{\theta}$.

THE DYNAMIC RELIABILITY ASSESSEMENT

For the first passage problem, the dynamic reliability defined given by $R = P\{X_l(\tau)\in\Omega_{\mathrm{s}}, \tau\in[0,T]\}$, where $P\{\cdot\}$ is the probability of the random event, Ω_{s} is the safe domain. It is equivalent to

$$R = P\{\sup_{\tau\in[0,T]} X_l(\tau)\in\Omega_{\mathrm{s}}\} = P\{Z_l\in\Omega_{\mathrm{s}}\} = \int_{\Omega_{\mathrm{s}}} p_{Q_l}(x,\tau=1)\mathrm{d}x \tag{5}$$

When the boundary is random, the dynamic reliability is $R = \int_{\Omega_{x_{\mathrm{B}}}}\left(\int_{\Omega_{\mathrm{s}}(x_{\mathrm{B}})} p_{Q_l}(x,\tau=1)\mathrm{d}x\right)p_{x_{\mathrm{B}}}(y)\mathrm{d}y$, in which $p_{x_{\mathrm{B}}}(\cdot)$ is the PDF of x_{B}, $\Omega_{x_{\mathrm{B}}}$ is the distribution domain of x_{B}, $\Omega_{\mathrm{s}}(x_{\mathrm{B}})$ means the safe domain is related to x_{B}.

NUMERICAL EXAMPLES

An 8-story shear-story frame (shown in Fig.1) with random stiffness, subject to El Centro earthquake with random peak ground acceleration (PGA) is studied. The masses of each story are 0.5, 1.1, 1.1, 1.0, 1.1, 1.1, 1.3, 1.2×10^5kg with the height h_1=4.0m, h=3.0m, the sections of the columns 500×500mm^2. The elastic modulus of the column is random with the mean 3.0×10^{10}Pa. $\boldsymbol{C}=a\boldsymbol{M}+b\boldsymbol{K}$, a=0.01Hz, b=0.005sec. The random parameters obey the truncated normal distribution with the coefficient of variation (COV) 0.2 for stiffness and 0.25 for PGA. The mean of the PGA=2.0m/s^2.
The dynamic reliability is defined by the top displacement. Part of the results are listed in Table 1, Table 2 and shown in Fig.2 and Fig.3. In Table 2 the threshold is assumed to be lognormal distributed with the mean 0.15 m and different COV, the EVD in Fig.2 and Fig.3 is the maximum absolute value of the top displacement, Fig.2 shows the computed EVD and the normal and lognormal distribution with the same second statistics and the Rayleigh distribution with the distribution parameter equal to the mean of the EVD. The comparison between the proposed method (PDEM) and the Monte Carlo simulation (MCM) shows that the proposed method is of accuracy. On the other hand, the proposed method is much time saving. Only 174 sec is needed with the proposed method while 13267 sec is needed with MCM.

Table 1 The dynamic reliability for different thresholds

Threshold	0.02 m	0.05 m	0.08 m	0.12 m	0.15 m
MCM	0.00391	0.24629	0.76743	0.97338	0.99033
PDEM	0.00456	0.25292	0.76761	0.97334	0.99010

Table 2 The dynamic reliability when the threshold is random

COV	5%	10%	20%	30%
MCM	0.98961	0.98871	0.98129	0.96652
PDEM	0.98979	0.98893	0.98319	0.96748

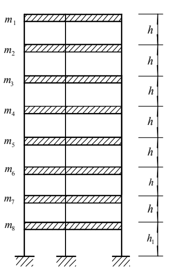

Fig.1 The 8-story frame

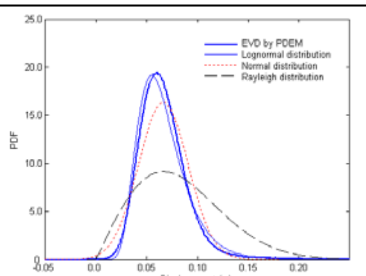

Fig.2 The EVD and widely used distributions

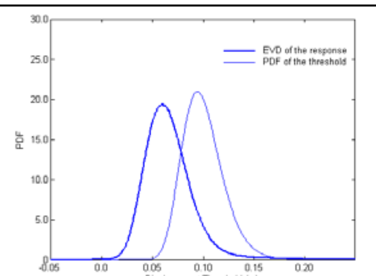

Fig.3 The EVD and the PDF of the threshold

CONCLUSIONS

An original method, based on the probability density evolution method, is proposed to compute the EVD. Naturally, the dynamic reliability is then easily obtained by the integration over the safe domain. The comparison with the Monte Carlo simulation demonstrates that the proposed method is of accuracy and efficiency.

References

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