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EFFECTIVE VISCOSITY OF AN INHOMOGENEOUS DILUTE SUSPENSION FLOWING ALONG A WALL

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Summary The classical result of Einstein for the effective viscosity of a dilute suspension of solid spheres in an arbitrary infinite Stokes flow is extended to account for the effect of a nearby wall. It is found theoretically that the presence of a wall amounts to a slip velocity for the suspension on a macroscopic scale. This slip velocity is obtained in term of the stresslet on a sphere, which is calculated analytically with the method of bipolar coordinates. Because of walls, the effective viscosity is reduced in a homogeneous suspension, in qualitative agreement with experiments. For a bounded suspension, the expression for the viscosity depends on the flow field, even in the first order in volume fraction. Moreover, the sensitivity of the effective viscosity to the inhomogeneity of the suspension is higher for a Poiseuille flow than for a shear flow.

INTRODUCTION

An essential theoretical result for the mechanics of unbounded dilute dispersions of spheres is the expression for the effective viscosity valid at order ϕ, where ϕ is the volume fraction (Einstein [1]). Later experiments performed in tubes showed a surprising result: the effective viscosity is reduced as compared with Einstein's result when the tube diameter decreases (see e.g. [2] and references herein). However, this effect was not fully explained theoretically. Later theories [3] [4] modelled this phenomenon for a homogeneous suspension in a shear flow: they showed that the suspension is slipping on walls on the macro-scale, thereby reducing the effective viscosity. The theory presented here accounts for the more general case of an inhomogeneous suspension in a Poiseuille flow between parallel planes. It will be shown that when the sphere radius is small compared with the distance between planes, considering the effects of the two walls separately provides a good approximation for the effective viscosity.

THEORY FOR THE EFFECTIVE VISCOSITY OF A SUSPENSION NEAR A WALL

Consider a dilute suspension of neutrally buoyant solid spherical particles with radius a flowing along a plane wall. The volume fraction is low: $\phi = (4\pi/3)\, a^3 n_0 << 1$, where n_0 is the number of particles per unit volume in the bulk of the suspension, that is far from the wall. Thus, hydrodynamic interactions between particles will be neglected. On the other hand, interactions between individual particles and the wall will be taken into account. The concentration distribution normal to the wall is general: $n(\mathbf{x}) = n_0 g_W(\mathbf{x}) = n_0 g(\mathbf{x}) H[d(\mathbf{x}) - a]$, where g is an arbitrary function and $d(\mathbf{x})$ is the distance from the particle center at $\mathbf{x}$ to the wall. The Heaviside function H account for the non overlapping of particles with the wall. Equations for the flow $(\mathbf{v}, p)$ valid in each point of a suspension of N particles centered at $\mathbf{x}^\beta$, $(\beta = 1 \cdots N)$ may be written as [5]:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0, \quad \mu_f \nabla^2 \mathbf{v} - \boldsymbol{\nabla} p = \sum_{\beta=1}^{N} \mathbf{f}(\mathbf{x}, \mathbf{x}^\beta, W) \tag{1}$$

using induced forces on the surfaces of the particles as defined by $\mathbf{f}(\mathbf{x}, \mathbf{x}^\beta, W) = \left[\boldsymbol{\sigma}^f \cdot \mathbf{n}\right] \delta(|\mathbf{x} - \mathbf{x}^\beta| - a)$, where $\boldsymbol{\sigma}^f$ is the stress tensor, $\mathbf{n}$ the normal outward unit vector on a sphere and δ is the Dirac function. The no-slip condition on the surfaces of particles is taken into account by these forces. The no-slip condition $\mathbf{v} = 0$ on the wall should be applied explicitly . Particles positions are random and effective properties of the suspension are considered here in the sense of ensemble averages. For a distribution (generalized function), this average $< . >$ is defined for a suspension of non interacting particles as:

$$(< q >, \varphi) = \int_V (q, \varphi)(\mathbf{x}_0) n(\mathbf{x}_0) d\mathbf{x}_0$$

where V is the volume of the whole suspension and φ denotes a test function. This problem is solved using a matched asymptotic expansions approach (like in [6]). In an inner region of order a close to the wall, equations (1) are averaged and the no-slip condition on the wall $< \mathbf{v} >= 0$ applies. In an outer region, far from the wall, induced forces average to zero for neutrally buoyant particles. Then averaging (1) simply gives the Stokes equations and their solution is the fluid velocity profile far from the wall. Only the boundary condition on the wall has to be replaced by a matching condition with the solution in the inner region. It is found that the resulting condition for the outer flow amounts to a slip velocity on the wall. Two different flow profiles are considered here, in view of possible experimental determinations of the effective viscosity: a shear flow and a Poiseuille flow. The Poiseuille flow can be written as the sum of a shear flow $v_x^\infty = k_1 z$ and a quadratic flow $v_x^\infty = k_2 z^2$. As an example, we present here the suspension flow profile as a result of matching for the case of the quadratic flow:

$$V_x(Z) = Z^2 + U_W \quad \text{where} \quad U_W = \frac{3\phi}{4\pi} \int_0^\infty Z \overline{f_x}(Z) dZ$$

All quantities are dimensionless: $V_x = \langle v_x \rangle / k_2 a^2$, $Z = z/a$ and $\overline{f_x}(Z) = \int_{\mathbf{x}\in\Re^3} G_W(Z_1) \widehat{f_x}(\mathbf{X}-\mathbf{X}_1, Z_1)\, d\mathbf{X}_1$, in which $G_W(Z_1) = G(Z_1)H(Z_1-1)$ (where $G(Z_1) = g(aZ_1)$) is the normalized distribution of particles and $\widehat{f_x}(\mathbf{X}-\mathbf{X}_1, Z_1)$ represents the dimensionless stress ($\widehat{f_x} = 1/(\mu_f k_2) f_x$) induced on the surface of a particle with center at the distance Z_1 from the wall. The slip velocity U_W may be expressed in term of the dimensionless stresslet S_{xz} on a particle:

$$U_W = U_0 + U_{HI} = U_0 - \frac{3\phi}{4\pi} \int_1^\infty G(Z_1)[S_{xz}(Z_1) - S_{xz}^\infty(Z_1)]dZ_1 \tag{2}$$

The stresslet was calculated analytically in bipolar coordinates and the integral determined with a 10^{-4} precision. We then obtained for a homogeneous suspension (viz. $G(Z_1) = 1$) in quadratic flow : $U_W^{\text{quadratic}} = 3.586\,\phi$. In a similar way, we found a slip velocity for a suspension in a shear flow: $U_W^{\text{shear}} = u_W^{\text{shear}}/(k_1 a) = 1.4419\,\phi$ which is in agreement with the result of Tözeren and Skalak[3]: $U_W^{\text{TS}} \simeq 1.45\,\phi$. The slip velocity was also calculated for typical inhomogeneous distributions of particles (with accumulation or depletion of particle near the wall) for both flow fields. It was found that the slip velocity is more sensitive to variations in the particle distribution for a quadratic flow than for a shear flow.
The effective viscosity of a inhomogeneous suspension sheared between parallel walls is, by taking in account each wall separately:

$$\langle \mu \rangle = \mu_f \left(1 + \frac{5}{2}\phi - \frac{2a}{h} U_W^{\text{shear}}\right) + O(\phi^2) \tag{3}$$

For the homogeneous case, it reduces to:

$$\langle \mu \rangle = \mu_f \left[1 + \frac{5}{2}\left(1 - 1.1535\frac{a}{h}\right)\phi\right] + O(\phi^2) \tag{4}$$

The effective viscosity of a suspension in Poiseuille flow is defined from the relationship between the pressure drop and the flux of the suspension. The result for an inhomogeneous suspension is:

$$\langle \mu \rangle = \mu_f \left[1 + \frac{5}{2}\phi - 6\frac{a}{h} U_W^{\text{shear}} + 6\left(\frac{a}{h}\right)^2 U_W^{\text{quadratic}}\right] + O(\phi^2) \tag{5}$$

and for the homogeneous case:

$$\langle \mu \rangle = \mu_f \left\{1 + \frac{5}{2}\left[1 - 3.4606\frac{a}{h} + 8.6065\left(\frac{a}{h}\right)^2\right]\phi\right\} + O(\phi^2) \tag{6}$$

Obviously, for $a/h \to 0$, Einstein result [1] is recovered in all cases. Brenner [7] found that a neutrally buoyant particle in a circular tube with radius R in Poiseuille flow induces a $O(a/R)^3$ pressure drop. His calculation uses the method of reflexions and therefore does not take into account the case where the particle is close to the wall. Nevertheless his result allows us to expect that the simultaneous effect of parallel walls on the effective viscosity in a two-dimensional Poiseuille flow will be at most of order $O[\phi(a/h)^3]$.

CONCLUSIONS

All terms involving a/h in the above results for the effective viscosity at order ϕ take into account exactly the interaction of individual particles with a wall. Since these terms are different for a shear flow and a Poiseuille flow, the effective viscosity of a wall bounded suspension is found to be dependent on the flow field, unlike the Einstein result for an unbounded suspension. The effective viscosity found for a suspension in Poiseuille flow is in qualitative agreement with experiments ([8], [9], etc. reviewed in [2]), that is it decreases when the distance h between the plates decreases. The high sensitivity of the effective viscosity to the distribution in particles in Poiseuille flow shows that it is essential to measure the particle distribution at the same time as the viscosity in order to estimate wall effects quantitatively and eventually compare with theory.

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