CONTINUITY CONDITIONS IN ELASTIC SHELLS WITH PHASE TRANSFORMATION

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Summary The six-field non-linear theory of elastic shells with the phase transformation of the material is developed. Equilibrium conditions are found from the variational principle of stationary total potential energy. New dynamic continuity conditions are derived at the movable singular surface curve modelling the phase interface. Particular forms of the continuity conditions at coherent and incoherent interface curves are given. The results are illustrated by an example of the phase transition in an infinite plate with a circular hole.

INTRODUCTION

Thin films made of shape-memory alloys like NiTi, NiMnGa, NiTiCu, or NiAl can considerably alter their shapes under appropriate external environmental changes. To model the mechanical behaviour of such thin films one can apply two-dimensional (2D) shell models consisting of a base surface endowed with appropriate fields modelling an additional microstructure. Then the notion of a movable surface curve separating the shell regions with different material phases is an appropriate tool for describing the interface of phase transition in the shell material.

A 2D model for single crystal thin films of martensitic materials based on the Cosserat membrane with one director field was proposed in [1]. However, kinematics of such a shell model is incomplete, for it does not contain the rotation about the director (the drilling rotation). The dynamic continuity conditions at the curvilinear phase interface of the shell are not available in the literature as well. The aim of this paper is to develop the complete non-linear theory of elastic shells with an account of occurrence of the phase transformation in the shell material.

WEAK FORMULATION OF THE BOUNDARY VALUE PROBLEM

Within the general non-linear theory of shells [2,3] the 2D local vector equilibrium equations and corresponding dynamic boundary conditions are formulated on the undeformed shell base surface M by an *exact* integration through the shell thickness of the corresponding 3D balance laws of continuum mechanics. The shell kinematics is then established through *energetically exact* dual fields in the 2D virtual work identity. In such a dynamically and kinematically exact shell theory the work-averaged translation vector $\boldsymbol{u}(\boldsymbol{x}) \in E$ and rotation tensor $\boldsymbol{Q}(\boldsymbol{x}) \in SO(3)$, $\boldsymbol{x} \in M$, are the only independent field variables of the boundary value problem (BVP).

In a two-phase elastic shell different material phases may appear on closed complementary subregions N_A and N_B of the deformed base surface $N = \chi(M)$ separated by the curvilinear phase interface D. For a continuous deformation χ we can introduce on M a singular curve $C = \chi^{-1}(D)$ separating $M_A = \chi^{-1}(N_A)$ and $M_B = \chi^{-1}(N_B)$. The position vectors of C and D are related by $\boldsymbol{x}_c(s) = \chi^{-1}(\boldsymbol{y}_c(s))$, respectively, where s is the arc length parameter along C.

The equilibrium BVP for shells with the phase transformation can be formulated in the weak form: Given the external force and couple vector fields $\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{c}(\boldsymbol{x})$ on M and $\boldsymbol{n}^*(s)$, $\boldsymbol{m}^*(s)$ prescribed along ∂M_f find a solution $(\boldsymbol{u}, \boldsymbol{Q}, \boldsymbol{x}_c)$ on the configurational space $C(M; E \times SO(3) \times E)$ satisfying the kinematic boundary conditions $\boldsymbol{u} - \boldsymbol{u}^* = \boldsymbol{0}$, $\boldsymbol{Q} - \boldsymbol{Q}^* = \boldsymbol{0}$ along $\partial M_d = \partial M \setminus \partial M_f$, such that for any kinematically admissible virtual vector fields $\delta\boldsymbol{u}$, $\boldsymbol{w} = ax(\delta\boldsymbol{Q}\boldsymbol{Q}^T)$, $\delta\boldsymbol{x}_c$ the following variational principle of stationary total potential energy is satisfied:

$$\delta I(\boldsymbol{u},\boldsymbol{Q},\boldsymbol{x}_c;\delta\boldsymbol{u},\boldsymbol{w},\delta\boldsymbol{x}_c) = 0\,, \quad I(\boldsymbol{u},\boldsymbol{Q},\boldsymbol{x}_c) = \iint_{M_A} W(\boldsymbol{u},\boldsymbol{Q})\,da + \iint_{M_B} W(\boldsymbol{u},\boldsymbol{Q})\,da - A(\boldsymbol{u},\boldsymbol{Q})\,. \qquad (1)$$

Here W is the 2D elastic strain energy density, and A is the potential of external loads such that $\delta A = \iint_M (\boldsymbol{f}\bullet\delta\boldsymbol{u} + \boldsymbol{c}\bullet\boldsymbol{w})\,da + \int_{\partial M_f} (\boldsymbol{n}^*\bullet\delta\boldsymbol{u} + \boldsymbol{m}^*\bullet\boldsymbol{w})\,ds$. As the stationarity conditions of I we obtain the known local equilibrium equations in M and the dynamic boundary conditions along ∂M_f, see [2,3].

DYNAMIC CONTINUITY CONDITIONS AT THE PHASE INTERFACE

Due to the phase transition of the shell material at the interface C, the principle (1) also requires that the following local dynamic continuity condition be satisfied (for details see [4]):

$$V[W] + [(\boldsymbol{N}\boldsymbol{\nu})\bullet\delta\boldsymbol{u}] + [(\boldsymbol{M}\boldsymbol{\nu})\bullet\boldsymbol{w}] = 0 \quad \text{along } C\,. \qquad (2)$$

Here $V = \delta\boldsymbol{x}_c\bullet\boldsymbol{\nu}$, $\boldsymbol{N} = \partial W/\partial\boldsymbol{E}$, $\boldsymbol{M} = \partial W/\partial\boldsymbol{K}$ are the resultant stress and couple tensors, $\boldsymbol{E}, \boldsymbol{K}$ are the shell strain and bending tensors, and $\boldsymbol{\nu}$ is the external unit normal to ∂M_A provided that the orientation of C coincides with that of ∂M_A. Besides, $[W] = W^+ - W^-$ is the jump of W at C with W^- and W^+ to be one-sided limits of W when the respective boundaries ∂M_A and ∂M_B of C are approached. Not all of the virtual fields V, $\delta\boldsymbol{u}^{\pm}$, and $\boldsymbol{w}^{\pm}$ in (2) are independent, in general.

At the *coherent* phase interface C both $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{Q}(\boldsymbol{x})$ are supposed to be continuous. In this case we have $[\boldsymbol{y}] = \boldsymbol{0}$, $[\boldsymbol{y}'] = \boldsymbol{0}$, $[\boldsymbol{Q}] = \boldsymbol{0}$, and $[\boldsymbol{Q}'] = \boldsymbol{0}$, where $(.)' \equiv d(.)/ds$. Then from the Maxwell theorem one can find the kinematic compatibility conditions $[\delta\boldsymbol{u}] + V[\boldsymbol{F}\boldsymbol{v}] = \boldsymbol{0}$ and $[\delta\boldsymbol{Q}] + V[(Grad_s\boldsymbol{Q})\boldsymbol{v}] = \boldsymbol{0}$ along C, with $\boldsymbol{F} = Grad_s\boldsymbol{y}$. This allows one to read off from (2) the following set of independent dynamic continuity conditions:

$$[\boldsymbol{N}]\boldsymbol{v} = \boldsymbol{0}, \quad [\boldsymbol{M}]\boldsymbol{v} = \boldsymbol{0}, \quad \boldsymbol{v}\bullet[\boldsymbol{C}_c]\boldsymbol{v} = 0 \quad \text{along } C. \tag{3}$$

Here $\boldsymbol{C}_c = W\boldsymbol{A} - \boldsymbol{N}^T\boldsymbol{F} - \boldsymbol{M}^T\boldsymbol{K}$, with $\boldsymbol{A}$ the surface metric tensor of M. The surface tensor $\boldsymbol{C}_c$ is an analog in shell theory of the Eshelby tensor used in continuum mechanics.

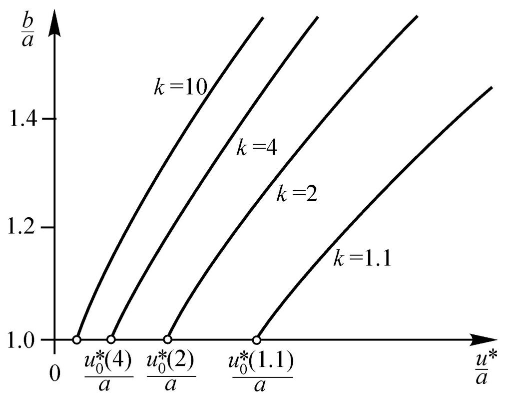
At the *incoherent* phase interface C, $\boldsymbol{y}(\boldsymbol{x})$ is still assumed to be continuous, but the continuity of $\boldsymbol{Q}(\boldsymbol{x})$ is allowed to be violated. It can then be shown that in this case $[\boldsymbol{Q}] \neq \boldsymbol{0}$ and $[\delta\boldsymbol{Q}] + V[(Grad_s\boldsymbol{Q})\boldsymbol{v}] \neq \boldsymbol{0}$. As a result, the condition (2) leads here to the following set of independent dynamic continuity conditions:

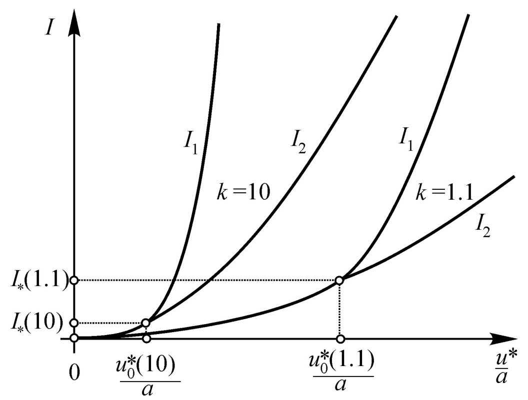
$$[\boldsymbol{N}]\boldsymbol{v} = \boldsymbol{0}, \quad \boldsymbol{M}^{\pm}\boldsymbol{v} = \boldsymbol{0}, \quad \boldsymbol{v}\bullet[\boldsymbol{C}_i]\boldsymbol{v} = 0, \text{ with } \boldsymbol{C}_i = W\boldsymbol{A} - \boldsymbol{N}^T\boldsymbol{F} \quad \text{along } C. \tag{4}$$

The conditions $(3)_{1,2}$ and $(4)_{1,2}$ express the balance of forces and couples at the phase interface C. The conditions $(3)_3$ and $(4)_3$ express the thermodynamic equilibrium of the material phases in the two-phase elastic shells. The latter conditions are necessary and sufficient for establishing the position vector $\boldsymbol{x}_c$ of C. The variational principle (1) and the dynamic continuity conditions (2)-(4) for elastic shells with phase transformation seem to be new in the literature.

EXAMPLE

The infinite plate with a central hole of radius a initially consisting of one material phase “B” is subjected to the uniform radial translation u^* of the hole boundary. Under the simple one-constants material laws of the phases defined by $W_{A,B} = C_{A,B}(E_{rr}^2 + \frac{1}{r^2}E_{\varphi\varphi})$ the transition to a new softer material phase “A” develops between the interface circle of radius b and the hole boundary. The solution exists only if $u^* > u_0^* = 2a/k$, with $k = C_B/C_A$. The left-hand Figure indicates the phase diagram for the values of b, and the right-hand Figure shows the values of the functional I of total potential energy calculated for the one-phase I_1 and two-phase I_2 solutions. I_2 is shown to be always lower than I_1.





Additional solutions of the phase transformation in a simply supported circular plate loaded by the surface pressure and in a semi-infinite circular cylinder loaded by the end couples were presented at the ICTAM2004.

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