DYNAMICAL SYSTEMS THEORY IN MATERIAL INSTABILITIES

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Summary The paper aims to present a unified method to deal with various material instability problems. As mathematical background the theory of dynamical systems is used, especially Lyapunov's stability definitions and elementary bifurcation theory. We follow continuum mechanics and its basic set of equations in describing solid bodies. By applying our methodology a few material instability phenomena are studied like strain localization or flutter, and a kind of mathematical interpretation can be obtained. There is also a possibility to explain the origin of mesh dependence in numerical studies of post-localization and the role of internal length in this context.

INTRODUCTION: BACKGROUND AND MOTIVATION

The definitions of material instability

The study of instability in elasto-plastic solid continua can be traced back to Drucker [1] and Hill [2] in the late fifties and has re-appeared in the famous work of Rice [3] on shear bands.

Since then there are a lot of papers pubished and several stability definitions are applied. These can be originated in either energy concepts [4] or kinematic conditions. The way of stability analysis is more or less determined by such definitions. For energy approaches the stability is defined as Lyapunov stability and the investigation performed is similar to the so called Lyapunov's direct method. In works based on kinematic conditions the study concentrates on wave speeds and the acoustic tensor. The sate of the solid body is considered to be stable, if all the eigenvalues c_i^2 (i.e. the squares of the wave speeds c_i) are strictly positive. Rice [3] introduces a classification for the loss of stability, which may be caused by the increase of loading. Firstly, the appearance of a zero eigenvalue is an instability being interpreted as the onset of localization (or divergence). Secondly, the material gets unstable, if the acoustic tensor has complex eigenvalues, which is called the flutter.

Dynamical systems, stability and bifurcation

In the theory of dynamical systems a system of autonomous differential equations

$$\dot{x} = g\left(x\right). \tag{1}$$

Assume that $x = 0$ is a solution of (1). Then the stability investigation of solution $x = 0$ can be performed by using Lyapunov's second method. Now a small perturbation is substituted $x = 0 + \Delta x$ into (1) and the linearized part Lx of $g\left(x\right)$ is studied.

The eigenvalues λ_i of operator L determine stability:

- if for all of them $\mathrm{Re}\lambda_i < 0$, the solution is stable;
- if for at lest one of them $\mathrm{Re}\lambda_i > 0$, the solution is unstable.

For the critical index k the stability boundary is $\mathrm{Re}\lambda_k = 0$. There are two basic ways of loss of stability:

- if $\mathrm{Im}\lambda_k = 0$, there is a static bifurcation;
- if $\mathrm{Im}\lambda_k \neq 0$, there is a dynamic bifurcation.

In this paper the Lyapunov stability definition is used, but we do not use energy functionals. Instead a set of partial differential equations is derived from the fundamental field equations of the classical continuum mechanics. These are interpreted as an infinite dimensional dynamical system for basic functions satisfying the boudary conditions. The Lyapunov's indirect method is applied for that generalized dynamical system.

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Assume that state S^0 of the solid body can be described by the values $v^0,\ \sigma^0,\ \varepsilon^0$ etc. and the stability of S^0 is studied. To explain the main steps of our investigation the study may be simplified to small deformation theory. Then the set of fundamental equations consists of the Cauchy equation of motion for the symmetric stress tensor σ

$$\rho\ddot{u} = \sigma\nabla, \tag{2}$$

the kinematic equation

$$\varepsilon = \frac{1}{2}\left(u \circ \nabla + \nabla \circ u\right), \tag{3}$$

and assume that the constitutive equation has the general form

$$0 = F\left(\varepsilon, \dot{\varepsilon}, \nabla^2\varepsilon, \nabla\varepsilon, \dot{\sigma}, \sigma\right). \tag{4}$$

After proper rearrangements and under some general conditions the set of equations (2), (3), (4) can be transformed into the field of the perturbation velocities

$$\Delta v = v - v^0.$$

Then a linear operator L is defined, which acts on the perturbation velocity field satisfying homogeneous boundary conditions. Note that L is a differential operator (represented by a set of partial differential equations). Now the investigations concentrates on the eigenvalues of that operator. The stability – instability conditions are the same as for dynamical systems (see the previous section). The main difficulty at this point is that the determination of the eigenvalues requires to solve a system of partial differential equations. This is almost impossible in a general case by analytic methods. Two approximate methods are applied instead. The one is a restriction into uniaxial case, the other is the use a specific set of perturbation functions, which are generally harmonic functions.

The first approach keeps the exact mathematics of the treatment, but in some cases leads to an over-simplification of the mechanical problem. The benefit is that we have a clear insight into what happens at the loss of stability. We could analyze the spectrum of the operator, its changes at the variation of the load and we may even look at the critical eigenspace of the critical eigenvalue. This point is closely related to the mesh sensitivity, mesh dependence appearing in numerical post-localization [5].

When a specific set of perturbation functions is applied we may consider more complicated mechanical problems, like nonlinear cases for example. Both material and geometrical nonlinearities (finite displacements) can be encountered.

APPLICATIONS, MAIN RESULTS

By using the method outlined above, we could obtain a few interesting observation on material instability phenomena. From the uniaxial investigations we find that the static or dynamic bifurcation seems to be a better classification of the loss of stability. Both divergence and flutter can be considered as special cases of one of them.

Our method enables us to study the role of the forms of the constitutive equations in the way of the loss of stability. We could show, how the inclusion of rate and first or second gradient dependence acts on the critical spectrum and eigenspace, and we are able to give a mathematical interpretation of the problematic behavior in numerical studies.

Especially, for the rate and second strain gradient dependent material we could show the origin of the nice post-localization behavior and the role of internal length. We can perform an analytical post-localization investigation for such material, and including material nonlinearity a so-called transcritical bifurcation can be found.

For large displacements we apply our method for the small pertubations of a state obtained by the equations of the large displacement theory. These perturbations are treated as in the linearized continuum mechanics. When a second order constitutive equation (a generalization of the second gadient dependent material) is used for large deformations, we get back a similar behavior as in small deformation case.

CONCLUSIONS

The theory of dynamical systems and Lyapunov's stability are useful mathematical tools for studying material instability problems. The benefits are quite obvious on the stability bondary when the way of the loss of stability is under consideration.

References

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