A STATE SPACE FORMALISM FOR PIEZOTHERMOELASTICITY OF FUNCTIONALLY GRADED MATERIALS

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Summary A state space formalism for electromechanical analysis of functionally graded materials (FGM) with rectilinear anisotropy and cylindrical anisotropy is presented. On the basis of the formalism, the piezothermoelastic solution to a problem may be determined analogously through its elastic counterpart. Viable schemes for solving the non-uniform state equation are proposed. Problems of electromechanical analysis of FGM can be treated in an elegant way.

STATE SPACE FORMULATION

The conventional approach to anisotropic elasticity is the Lekhnitskii or Stroh formalism. For piezothermoelasticity the usual approach is to extend the Stroh formalism to include the piezoelectric effects. Extension of the Stroh formalism to piezothermoelasticity of FGM is unwieldy in formulation and restrictive in application. Herein we present a general formalism and solution approach for electromechanical analysis of FGM. The piezoelectric materials considered possess rectilinear anisotropy or cylindrical anisotropy of the most general kind. The novelty of the formalism lies in that the 3D equations of piezothermoelasticity are represented in full by a state equation and an output equation in which only a displacement vector, a stress vector, and six sub-matrices that characterize the material properties appear. This is achieved through grouping the field variables and selecting the state vector judiciously. In addition, the equations of piezothermoelasticity in the state space bear a remarkable resemblance to their elastic counterparts. As such, the piezothermoelastic solution to a problem may be obtained in an analogous manner through the corresponding elastic solution. The formalism brings in matrix algebra in the solution. When dealing with FGM, the material inhomogeneity renders the system non-uniform; it is very difficult to obtain an analytic solution to the state equation in general. Viable schemes for solving the non-uniform state equation are suggested. The schemes are useful for problems both in Cartesian coordinates and in cylindrical coordinates.

FGM with Rectilinear Anisotropy

The fundamental equations of piezothermoelasticity are given in [1]. To avoid dealing with individual field variables and many of the material constants, we make judicious grouping of the field variables and partitioning of the constitutive matrices. On the basis of the derivation in [2-4] for homogeneous anisotropic elastic and piezoelectric materials, the basic 3D equations in Cartesian coordinates can be reformulated into a state equation and an output equation as follows:

$$\frac{\partial}{\partial x_2}\begin{bmatrix}\mathbf{u}\\ \tau_2\end{bmatrix}=\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{11}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \tau_2\end{bmatrix}+\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{bmatrix}\mathrm{T}-\begin{bmatrix}\mathbf{0}\\ \mathbf{F}\end{bmatrix}+\frac{\partial^2}{\partial t^2}\begin{bmatrix}\mathbf{0}\\ \mathbf{Ku}\end{bmatrix} \tag{1}$$

$$\tau_2=\begin{bmatrix}\mathbf{L}_1 & \mathbf{L}_2\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \tau_2\end{bmatrix}-\mathbf{b}_3\mathrm{T} \tag{2}$$

The notations similar to those in [2-4] are used. x_2 is considered to be a distinct axis; the state vector consists of $\mathbf{u}=[u_1\ u_2\ u_3\ \varphi]^{\mathbf{T}}$ and $\tau_2=[\sigma_{12}\ \sigma_{12}\ \sigma_{12}\ D_2]^{\mathbf{T}}$, the vector τ_1 contains the remaining stress components and electrical displacements; $\mathbf{D}_{ij}$ are linear differential operators of x_1, x_3 and the material constants. For FGM the system matrix in (1) is spatial dependent and the state equation is non-uniform with variable coefficients.

FGM with Cylindrical Anisotropy

The state equation and output equation in cylindrical coordinates take the form

$$\frac{\partial}{\partial \mathrm{r}}\begin{bmatrix}\mathbf{u}\\ \mathrm{r}\tau_{\mathbf{r}}\end{bmatrix}=\begin{bmatrix}\mathbf{L}_{11} & \mathbf{L}_{12}\\ \mathbf{L}_{21} & \mathbf{L}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathrm{r}\tau_{\mathbf{r}}\end{bmatrix}+\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{bmatrix}\mathrm{T}-\begin{bmatrix}\mathbf{0}\\ \mathbf{F}\end{bmatrix}+\frac{\partial^2}{\partial t^2}\begin{bmatrix}\mathbf{0}\\ \mathbf{Hu}\end{bmatrix} \tag{3}$$

$$\mathrm{r}\tau_{\mathbf{s}}=\begin{bmatrix}\mathbf{L}_1 & \mathbf{L}_2\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathrm{r}\tau_{\mathbf{r}}\end{bmatrix}-\mathbf{a}_3\mathrm{T} \tag{4}$$

where r is considered to be a distinct axis; $\mathbf{u}=[u_r\ u_\theta\ u_z\ \varphi]^{\mathbf{T}}$, $\tau_{\mathbf{r}}=[\sigma_{\mathbf{r}}\ \sigma_{\mathbf{r}\theta}\ \sigma_{\mathbf{rz}}\ D_r]^{\mathbf{T}}$, $\tau_{\mathbf{s}}$ contains the remaining components of the field variables; $\mathbf{L}_{ij}$ are linear differential operators of θ, z and the material constants.

Equations (1) to (4) bear a remarkable resemblance not only in themselves but also to their elastic counterparts [2, 3]. As a result, the piezothermoelastic solution of a problem may be determined from the corresponding elastic solution by analogy and correspondence. A proper replacement of the corresponding matrices in the elastic solution produces the piezothermoelastic solution. When the formalism is applied to plane problems of homogeneous materials, the solution of the homogeneous equations leads naturally to the eigenrelation and the octet equation based on the Stroh formalism [4].

Equations (1) and (3) are not easy to solve in general except for special classes of FGM, such as a power-law distribution of the inhomogeneity. A formal solution may be obtained by using successive integration. Yet it is difficult to carry out the integration and assess the convergence. Herein we propose two solution schemes.

Piecewise-Constant Approximation

The basic idea is to approximate the inhomogeneity of the FGM by piecewise-constant functions so that the state equation in the sub-regions becomes uniform and solvable analytically. Continuity conditions at the jumps introduced by the piecewise-constant approximation are satisfied by means of the transfer matrix. In the case of a FGM plate (or a radially inhomogeneous FGM cylinder), the scheme amounts to approximating the FGM body by a system composed of homogeneous layers (or coaxial homogeneous cylinders). It applies to a laminated system in an obvious way. The scheme has been applied to problems of circular cylinders and tubes of FGM and laminated composites [5].

Power Series Approximation

The scheme is well suited when the inhomogeneity of the FGM is describable by polynomials. For illustration, let us consider FGM with radial inhomogeneity. Suppose, upon expressing the dependence of θ and z in some way (such as Fourier series representation), that (3) reduces to

$$\frac{d}{dr}\mathbf{X}(r)=\mathbf{A}(r)\mathbf{X}(r)+\mathbf{P}(r) \tag{5}$$

The dependence of the material property on r renders the system matrix $\mathbf{A}(r)$ a polynomial given by

$$\mathbf{A}(r)=\mathbf{A}_0+\mathbf{A}_1 r+\mathbf{A}_2 r^2+\mathbf{A}_3 r^3 \tag{6}$$

where $\mathbf{A}_i$ are known constant matrices.

The homogeneous solution of (5) can be determined by assuming a power series representation of $\mathbf{X}(r)$:

$$\mathbf{X}(r)=\sum_{n=0}^{\infty}\mathbf{X}_n r^n \tag{7}$$

where $\mathbf{X}_n$ are unknown vectors. A substitution of (7) in (5) leads to the following recursive relation

$$(n+1)\mathbf{X}_{n+1}=\mathbf{A}_0\mathbf{X}_n+\mathbf{A}_1\mathbf{X}_{n-1}+\mathbf{A}_2\mathbf{X}_{n-2}+\mathbf{A}_3\mathbf{X}_{n-3}, \quad \mathbf{X}_{-1}=\mathbf{X}_{-2}=\mathbf{X}_{-3}=0, \quad n=0,1,2,\cdots \tag{8}$$

The relation allows one to express the higher-order terms of $\mathbf{X}_n$ in terms of the lower-order terms, and $\mathbf{X}(r)$ in turn can be expressed in terms of $\mathbf{X}_0$ as

$$\mathbf{X}(r)=\sum_{n=0}^{\infty}\mathbf{X}_n r^n=\mathbf{M}(r)\mathbf{X}_0 \tag{9}$$

Now $\mathbf{M}(r)$ plays the role of the transfer matrix. With the homogeneous solution so determined, the particular solution of (5) can be easily found using a standard method of matrix algebra. The unknown in $\mathbf{X}_0$ can be determined from the boundary conditions on the cylindrical surfaces $r = constant$ for a specific problem.

SOME REMARKS

The state space formalism is elegant in formulation and systematic in operation. It has been used for treating the generalized plane strain and generalized torsion of an elastic body [6] and a piezoelectric body [7], where focuses were on formulations and solutions of specific problems. A number of exact solutions have been obtained with relative ease using the approach. For more general problems Hamiltonian characteristics of the system may be used to advantage. For piezothermoelasticity of FGM the material inhomogeneity, in addition to electromechanical coupling and material anisotropy, leads to a challenging mathematical problem. The state space formalism makes treatment of the problem less formidable. It appears that the electromechanical analysis of FGM is best viewed and treated in the state space setting.

References

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