CONTACT MECHANICAL ANALYSIS OF ELASTIC MULTIBODY STRUCTURES

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Summary A theory of multibody structures of elastic blocks interconnected by plane dry joints with Coulomb friction is developed by basing the analysis on contact mechanics. A general theory of voussoir arches and segmental beams is developed.

This analysis is based on the contact mechanics of elastic blocks with approximately rectangular longitudinal section subjected at their end faces to normal force N, moment M and shear force Q with friction coefficient f at the plane joints. It is then possible to relate corresponding generalized deformations, translations v_x, v_z and rotations ω_y, linearly to N, Q and M. Based on these statical and kinematical concepts and their interrelations a mechanics of assemblages of elastic units in contact is developed and the effects of various structural factors are discussed.
If the blocks are rigid, the kinematics at collapse is determined by discontinuities at joints, mutual translations v_{xh}, v_{yh} and dito rotations ω_{yh}, which define linear gap deformations γ_x, γ_z with rotation axes being tangent to the cross section A or constant contact sliding γ_z over the whole of A. In both cases contact sliding $\gamma_z \neq 0$ requires

$$| q | = | Q/(fN) | = 1 \tag{1}$$

The corresponding load P_c can uniquely be determined only if friction coefficient f = tanφ = 0 or ∞. In these cases the set of collapse loads P_c represent the lateral surface of uniquely determined convex cones $E_{\varphi=0}$ (shear stress τ = 0) and $E_{\varphi=\pi/2}$ (slip $\gamma_z = 0$). In the intermediate range 0 < φ < π/2 the corresponding E(φ) cannot be uniquely determined. We introduce a combined friction f = tan(ρ+β), where β characterizes friction induced by the inclination tanβ of the conical asperities on the contact plane and ρ represents the dissipative friction on the inclined slopes. At given φ = ρ+β the cone E(ρ,φ-ρ) represents a set of inclusions with extreme member E(0,φ) ⊃ E(ρ,φ-ρ) and to every E(ρ,β) corresponds its orthogonal polar cone E*(ρ,β) = Ξ(β) (Fig. 1b) that comprises the set of desintegrating velocities.
If the voussoirs are elastic we can separate the deformations of the blocks from those of the joints by linearization of the crack width to $\bar{\gamma}_{xh}(z) = v_{xh} + z\omega_{yh}$ and constant $\bar{\gamma}_{zh}$ over the entire surface A. This actually represents an extension to cracks of Reissner's linearizations in his plate theory. The modified stress energy enlarged with the slip work of the joints of the arch is thus

$$W''(\sigma) = \frac{1}{2}\int_L \left(\frac{N^2}{EA} + \frac{M^2}{EI} + \frac{\xi Q^2}{GA} \right) ds + \frac{1}{2}\sum_{i=1}^{n} \left(\frac{|N^i \bar{\gamma}^i_{xh}(e)|}{EA} + \frac{|Q^i \bar{\gamma}^i_{zh}|}{GA} \right) \tag{2}$$

Here $\bar{\gamma}_{xh}{}^i(e) = -v_{xh}{}^i(e)$ represents the interpenetration at the eccentricity e = km of force N (k is the kern point distance; for a rectangular cross section k = d/6) and $\bar{\gamma}_{zh}{}^i$ denotes the generalized shear deformation (Fig. 1a)

$$v^i_{zh} = \bar{\gamma}^i_{zh} = \int_A \frac{S_y \gamma_z}{I} dA \quad ; \quad \text{with rectangular cross section } A = bd : \ \bar{\gamma}^i_{zh} = \frac{3b}{2A}\int_d \left(1 - (2z/d)^2\right)\gamma_z dz \tag{3}$$

where the integrals are extended over the area $A_\gamma = A - A_c + A_s$; A_c is the contact area and A_s denotes the area of contact sliding $A_s \subset A_c$. In order to make the v_{xh} independent of Q the joint plane must be a plane of symmetry with regard to the internal forces N and Q. This warrants that the contact area for combination (N,Q) doesn't differ from that for pure N and the nonelastic v_{zh} can be included in the elastic scheme. Thus we obtain

$$v^i_{xh} = \frac{|N^i| d}{EA}\eta_h(m^i); \quad \omega^i_{yh} = \frac{|N^i| d}{EAk}\alpha_h(m^i); \quad v^i_{zh} = \frac{Q^i d}{GA}\kappa_h(m^i, q^i); \quad |v^i_{xh}(e)| = \frac{|N^i| d}{EA}\delta_h(m^i) \tag{4}$$

where $v_{xh}{}^i$ only depends on the crack volume $V_h{}^i = \int \gamma_x{}^i dA$ and (because of the impenetrability) stretches the centroid axis even if the crack doesn't extend to the centroid of the cross section. The quantity q = Q/|fN| expresses the fullness of frictional contact at the joint –1 ≤ q ≤ 1 (Eqn. 1). Furthermore there applies

$$\delta_h = m\alpha_h - \beta_h \quad ; \quad \partial\delta_h / \partial m = 2\alpha_h \tag{5}$$

If the compressive force N acts at a point on the contour of the A with extreme value $| m |_{max}$ the values of α_h, β_h, δ_h and κ_h → ∞. Similarly if | q | → 1 then κ_h → ∞ inducing unbounded translation γ_z on the entire contact area A_c of the joint. It

should be observed that the rotation axis of the end faces doesn't coincide with the zero-line of the σ_x stresses as in the corresponding rigid-plastic case, because the hinge is situated inside the compressed region (Fig. 1a).
The above results concerning symmetric joints of equal blocks can be extended also to curved voussoirs of varying length and depth and with unsymmetric joints by retaining equal plane contact faces A_c and equal normal forces N with the same point of action and equal shear forces at the joint. Thus for a joint of two ashlars (1) and (2) with rectangular cross sections $A_1 = bd_1$, $A_2 = bd_2$ the parameters of the joint are

$$\alpha_h \approx ½\,(\alpha_{h1}(m_1)+\alpha_{h2}(m_2))\,; \quad \beta_h \approx ½\,(\beta_{h1}(m_1)+\beta_{h2}(m_2))\,; \quad \kappa_h \approx ½\,(\kappa_{h1}(q_1,m_1)+\kappa_{h2}(q_2,m_2)) \qquad (6)$$

Corresponding approximation of of α_h , β_h and κ_h are determined for a collection of typical joints of beams, columns, plates and vaults.
Using formulae (2), (4) and (5) a solution of the redundant forces, thrust H, shear Y and moment Z at the elastic centroid of an arch with fixed supports is obtained by minimum condition of $W(\sigma)$ and an iterative solution process. Thus for the horizontal thrust we obtain the equation

$$H\int_L\left(\frac{z^2}{i^2}+\cos^2\vartheta+\frac{E}{G}\xi\sin^2\vartheta\right)\frac{ds}{EA}=\int_L\left(\frac{M^o z}{i^2}+N^o\cos\vartheta+\frac{E}{G}\xi Q\sin\vartheta\right)\frac{ds}{EA}+\sum_{i=1}^{n}\left(\frac{|N|d}{EA}(\eta_h\cos\vartheta+\frac{z}{k}\alpha_h)+\frac{Q}{GA}\bar{\gamma}_z\sin\vartheta\right) \qquad (7)$$

In the formulas (2) and (7) the integrals on the right hand side express the share of the monolithic arch, whereas the sums represent the share of the corresponding rigid body assemblage with dry joints.
A comprehensive representation provides the stiffness surface $F(\Delta)$ and the cone of load displacements Ξ (Fig. 1b). The effects of various factors (the number of blocks, the slenderness of the arch, the value of the friction coefficient) on the stiffness surface and the cone of load displacements of the structure are presented.
The voids γ_x developing at the joints may have a remarkable stabilizing effect at dynamic actions. The kinetic energy of an impact is mainly balanced by the increased potential energy of gravity induced by the vertical components of the crack volumes which many times exceeds the share of the strain energy (Fig. 1c).

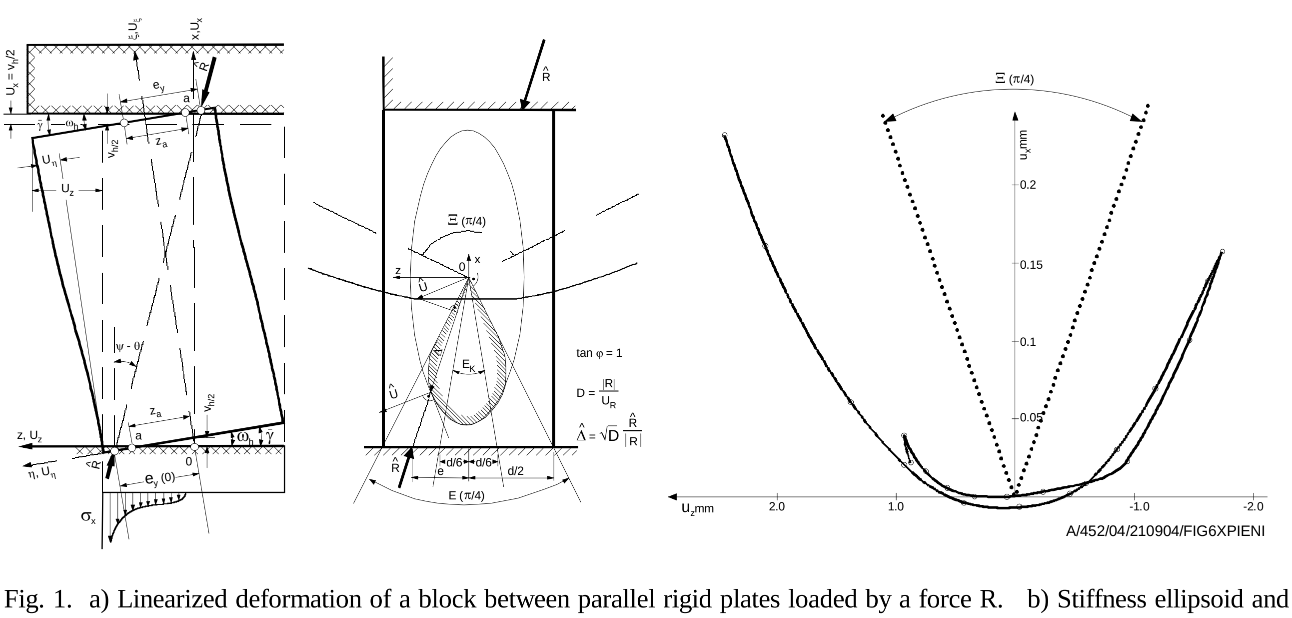

Fig. 1. a) Linearized deformation of a block between parallel rigid plates loaded by a force R. b) Stiffness ellipsoid and stiffness surface $F(\Delta)$ (shaded) of a block with the cone of monolithic kern E_k, the cone of stability $E(\pi/4)$ and the truncated cone of displacements Ξ. c) Staggering restitution u_x , u_z of the upper plate from imposed horizontal dislocation (test).

References

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