PROBABILITY PHENOMENA IN PERTURBED DYNAMICAL SYSTEMS

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Abstract We discuss probability phenomena associated with passages through separatrices and passages through resonances in perturbed dynamical systems. The theory which describes these phenomena has applications in different problems including problems of capture of satellites into resonances, acceleration of charged particles, chaotic advection of impurities.

1. Introduction

If in a deterministic dynamical system a small variation of initial data produces a big variation of dynamics, then the behavior of such a system can be treated as a random one. This non-rigorous assertion known as a principle “small causes and big effects” is in the basis of the theory of deterministic chaos. Remarkably, such quasi-random behavior exists also in the systems which differ by an arbitrarily small perturbations from systems with very simple (periodic, quasi-periodic) dynamics. Different types of perturbed dynamics have certain probabilities. Analysis of long-term dynamics leads to random walk problems.

We discuss probability phenomena associated with passages through separatices and passages through resonances in perturbed dynamical systems.

2. Origin of Probability Phenomena in Perturbed Systems

We consider probability phenomena in systems which can be described by ordinary differential equations (ODE). These are deterministic systems: for ODE under consideration on a given time interval the solution is uniquely defined by initial data and depends smoothly on these data. The origin of probability phenomena in such systems is associated with sensitive dependence of dynamics on initial data. To note this sensitive dependence and probability phenomena one should observe behavior of the system on a long enough time interval. Let initial data be known with an accuracy δ and behavior of a system be observed on time interval of length T. For given T one can choose δ small enough such that any δ-variation of initial data almost does not change the solution of the ODE on the time interval of the length T. But for given δ it can happen that for T big enough a δ-variation of initial data changes the solution over time T considerably (by a value which does not depend on δ). In this case knowledge of initial data with accuracy δ is not enough to predict behavior of the system over the time T. From more formal viewpoint this is the situation when limits as $\delta \to 0$ and as $T \to \infty$ do not commute. Probabilistic approach is useful in this situation.

We will consider systems which can be described by ODE of the form

$$\dot{y} = v(y) + \varepsilon w(y, \varepsilon), \quad y \in R^l, \quad 0 < \varepsilon \ll 1. \tag{1}$$

Here y is a vector of phase variables, ε is a small parameter. (In Eq. (1) instead of $y \in R^l$ often one should consider $y \in M^l$, where M^l is an l-dimensional smooth manifold, say cylinder, sphere or torus.) For $\varepsilon = 0$, we get *the unperturbed system.* Term εw in Eq. (1) is called *the perturbation.* System (1) is called *perturbed system.*

To note a considerable (of order 1) effect of the perturbation one should observe behavior of solutions of the system (1) on time interval whose length tends to ∞ as ε tends to 0. We will consider cases when such time interval should be at least of the length of order of $1/\varepsilon$. If limits as δ (accuracy of knowledge of initial data) tends to 0 and as ε tends to 0 do not commute, probabilistic approach is useful.

The *averaging method* is one of the methods of approximate description of dynamics of perturbed systems on long time intervals.

Assume that for system (1) the unperturbed ($\varepsilon = 0$) system is integrable. There are different definitions of integrability (see, e.g. [5]). But we here will call a system *integrable* in some domain of the phase space if the following holds. This domain up to a residue set of a measure zero is smoothly foliated by invariant tori (in particular case, by invariant closed curves), and in a neighborhood of every such torus one can introduce new variables (x, φ), $x = (x_1, \dots, x_n) \in R^n$, $\varphi = (\varphi_1, \dots, \varphi_m) \in T^m$modd 2π, $n + m = l$, such that the equations of the motion in the new variables have the form

$$\dot{x} = 0, \quad \dot{\varphi} = \omega(x). \tag{2}$$

Therefore, components of the vector x are first integrals of the unperturbed system. Variable x enumerate invariant tori of the unperturbed system. Components of vector φ are angular variables (phases) on these tori; ω is a vector of frequencies of motion on tori.

Transformation of variables $y \mapsto (x, \varphi)$ transforms perturbed system (1) into the form

$$\dot{x} = \varepsilon f(x, \varphi, \varepsilon), \quad \dot{\varphi} = \omega(x) + \varepsilon g(x, \varphi, \varepsilon). \tag{3}$$

System (3) is called *system with rotating phases*. Variables x and φ in it are called *slow variables* and *fast variables* (*fast phases*) respectively.

For approximate description of the evolution of the slow variables x the averaging method (see, e.g. [5, 9]) prescribes to average rate of changing of x over fast phases φ. In other words, one should replace Eq. $(3)_1$ with the *averaged equation* (or the *averaged system*)

$$\dot{x} = \varepsilon F(x), \quad F(x) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \dots \int_0^{2\pi} f(x, \varphi, 0) d\varphi_1 \dots d\varphi_m. \tag{4}$$

The recipe of the averaging method is not always applicable for approximate description of dynamics of a perturbed system. In particular, obstacles for applicability of the averaging method are separatrices and resonances.

A) Separatrices. Assume that some domain in the phase space of the unperturbed system up to a residue set of a measure zero is smoothly foliated by invariant closed curves. On the residue set this foliation is singular. The residue set consists of points of equilibrium and *separatrices* passing through them. Examples of separatrices are shown in Figs. 2, 6. Separatrices divide the phase space into several regions. In every one of these regions equations of the unperturbed motion can be written in the form of Eq. (2), and equations of the perturbed motion can be written in the form of Eq. (3). On separatrices these equations have singularities. In the neighborhood of separatrices the equations of the unperturbed (respectively, perturbed) motion can not be written in the form of Eq. (2) (respectively, Eq. (3)). Under the action of perturbation phase points that initially were at the distance of order 1 from the separatrix of unperturbed system can arrive to the separatrix during the time of order $1/\varepsilon$.

B) Resonances. Assume that perturbed system can be written in the form of the system with rotating phases (3). Consider for simplicity two-frequency case: $m = 2$, $\varphi = (\varphi_1, \varphi_2)$, $\omega = (\omega_1, \omega_2)$. Invariant torus of unperturbed system is said to be *resonant* (respectively, *nonresonant*), if the ratio of unperturbed frequencies of the motion on this torus $\omega_1(x)/\omega_2(x)$ is a rational (respectively, an irrational) number.

On a resonant torus all unperturbed trajectories are closed curves. On a nonresonant torus every unperturbed trajectory fills the torus densely. Averaging in Eq. (4) is performed over invariant torus of unperturbed system. On a resonant torus an unperturbed trajectory is a closed curve. So, averaging over torus may be not adequate for description of dynamics near a resonance.

Assume that $\partial(\omega_1/\omega_2)/\partial x \neq 0$. For given integer nonzero numbers k_1, k_2 the relation $k_1\omega_1(x) + k_2\omega_2(x) = 0$ defines in the space of the slow variables x a surface which is called the *resonant surface* for (k_1, k_2)-resonance. Under the action of a perturbation phase points that initially were at the distance of order 1 from this resonant surface can arrive to it during the time of order $1/\varepsilon$.

The obstacles to the averaging method play an important role in the origin of probabilistic behavior in perturbed systems. In absence of these obstacles the sensitive dependence on initial data and probabilistic behavior appear first (in time) in dynamics of fast variables (phases). However, from the viewpoint of applications more important is dynamics of slow variables. Averaging method ignores dynamics of fast phases by averaging it out. At separatrices and at resonances the indeterminacy in fast phases is transformed into indeterminacy in slow variables.

There are probability phenomena associated with passages through separatrices and with passages through resonances in different classes of systems: in general systems (i.e. in systems without any specific structure), in Hamiltonian systems with slowly varying parameters and in Hamiltonian systems with slow and fast variables, in volume-preserving systems. For each of these classes of systems there is corresponding theory of probability phenomena.

3. Passages through Separatrices

We start with a basic example of probabilistic scattering on separatrix.

Example 1 (see [3]) *Consider motion of a material point in one-dimension in a double-well potential $V(q)$ (Fig. 1) in presence of a small friction $\varepsilon f(\dot q, q)$.*

Equations of motion have the form

$$\dot p = -\frac{\partial V(q)}{\partial q} + \varepsilon f(p, q),\ \dot q = p. \tag{5}$$

Consider motion of points whose initial energy is higher than the energy of potential hump separating the wells. Under the action of friction almost every such point after some time (typically, after the time of order $1/\varepsilon$) will be trapped into one of the potential wells, A or B. But into exactly which one?

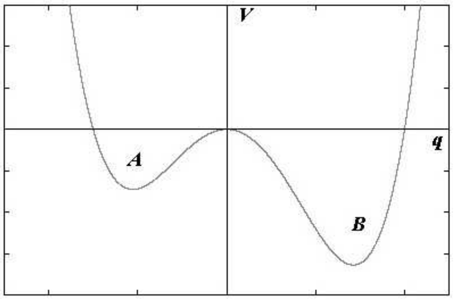

Figure 1. Double-well potential.

The phase portrait of the unperturbed ($\varepsilon = 0$) system is shown in Fig. 2. The separatrices l_1 and l_2 divide it into domains G_1, G_2, G_3. On the phase portrait of the perturbed ($\varepsilon > 0$) system (Fig. 3) the initial data that lead to trapping into different wells alternate.

If initial data are known with accuracy δ, then for $\varepsilon \ll \delta$ we cannot predict the final result of the evolution. So, in the limit as $\varepsilon \to 0$ the deterministic approach to the problem fails. However, it is possible to treat trapping into one well or the other as random events and calculate probabilities of these events.

The probabilities $P_1(M_0)$ and $P_2(M_0)$ of the trapping of the initial point $M_0 = (p_0, q_0)$ into potential well A (or, which is the same, into domain G_1) and, respectively, into potential well B (or, which is the same, into domain G_2) can be defined as follows. Let $U^\delta(M_0)$ be δ-neighborhood of M_0 in the phase plane. Let $U_i^{\delta,\varepsilon}(M_0)$ be set of points from $U^\delta(M_0)$ that will be trapped into domain G_i, $i = 1, 2$. Then, by definition

$$P_i(M_0) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{\operatorname{mes} U_i^{\delta,\varepsilon}(M_0)}{\operatorname{mes} U^\delta(M_0)}. \tag{6}$$

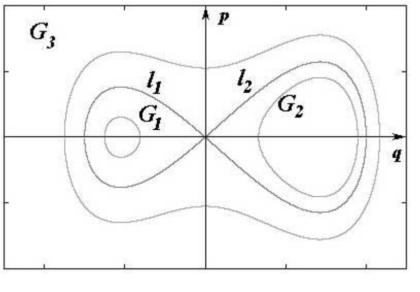

Figure 2. Phase portrait of the motion in the double-well potential.

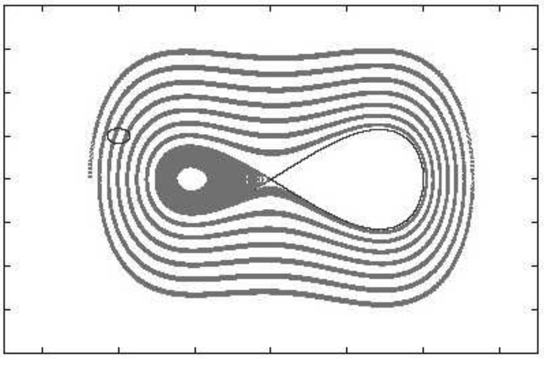

Figure 3. Phase portrait of the motion in the double-well potential in presence of a small friction.

Here $\mathrm{mes}\,(\cdot)$ *is an area in the phase plane. It turns out that the probabilities of trapping into different wells exist and are computed by the formulas*

$$P_i(M_0) = \frac{\Theta_i}{\Theta_1 + \Theta_2}, \quad i = 1,2, \quad \Theta_\nu = -\oint_{l_\nu} pf(p,q)dt, \quad \nu = 1,2. \quad (7)$$

The integrals in Eq. (7) are taken over separatrices l_ν *of the unperturbed system in the unperturbed motion. The probabilities do not depend on the initial point* M_0 *and are defined by the values of* $f(p,q)$ *on the critical energy level corresponding to separatrices.*

In the problem under consideration it is usually of interest to consider evolution of the energy $h = p^2/2 + V(q)$ *with time. Far from separatrices of the unperturbed system this evolution is approximately described using averaging: one can rewrite the equations of the perturbed motion in the standard form of Eq. (3) and then average the equation for slow variable over the fast phase. As a slow variable we can use the unperturbed energy* h*. How does one describe the passage through the separatrices? Fig. 4 shows three solutions of the averaged system. Value* $h = 0$ *corresponds to the unperturbed separatrix. Variable* τ *is the slow time:* $\tau = \varepsilon t$*. The solution* $h_3(\tau)$ *in the domain* $G_3 = \{h > 0\}$ *starts for* $\tau = 0$ *with a value equal to the energy of the initial point* M_0*, and arrives to the separatrix* $(h = 0)$ *at some* $\tau = \tau_*$*. We glue to this solution the solutions* $h_1(\tau)$ *and* $h_2(\tau)$ *in domains* G_1 *and* G_2 *respectively starting at* $\tau = \tau_*$ *on the separatrix:* $h_{1,2}(\tau_*) = 0$*. It turns out that the motion of majority of the phase points trapped into the well* A *(respectively,* B*) at time intervals of the length* $\sim 1/\varepsilon$ *is approximately described by gluing of the solutions* h_3 *and* h_1 *(respectively,* h_3 *and* h_2*). The exception is formed by the points that pass very close to the saddle point* C *and because of this are*

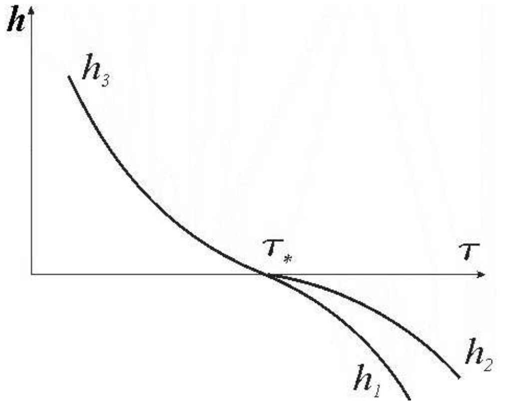

Figure 4. Evolution of the energy for the averaged equation in a case of separatrix crossings.

moving near it for a long time; the measure of the set of these points tends to zero as $\varepsilon \to 0$. We can say that with probability P_1 the motion is described by gluing of h_3 and h_1, and with probability P_2 by gluing of h_3 and h_2.

An analogous loss of determinacy occurs if the friction is replaced by a slow variation of the potential in time leading to a fall into one well or the other (for example, if a point is moving along a curve with two minima in a slowly increasing gravity field).

A general situation in which such phenomena arise can be described as follows. We have the system of equations

$$\dot{p} = -\frac{\partial H}{\partial q} + \varepsilon f_1, \quad \dot{q} = \frac{\partial H}{\partial p} + \varepsilon f_2, \quad \dot{\lambda} = \varepsilon f_3, \tag{8}$$

where $0 < \varepsilon \ll 1$, $(p, q) \in R^2$, $\lambda \in R^k$, $H = H(p, q, \lambda)$, $f_i = f_i(p, q, \lambda, \varepsilon)$. The functions f_i are assumed to be smooth enough. The unperturbed ($\varepsilon = 0$) system for p, q is a Hamiltonian one; its Hamiltonian function H depends on parameter λ. For definiteness we shall assume that the phase portrait of the unperturbed system for all λ has the same form in Fig. 2 (but other portraits can also be considered, in which there are nondegenerate saddles joined by the separatrices). The separatrices l_1 and l_2 divide the unperturbed phase plane into three domains: G_1, G_2, G_3 (see Fig. 2). Under the influence of the perturbation the points from the domain G_i cross the separatrix and are trapped in one of the domains G_j, $j \neq i$. Capture by one domain or the other thus should be treated as a random event. The probabilities of these events for an initial point $M_0 = (p_0, q_0, \lambda_0)$ are defined by the relation (6).

We shall assume the Hamiltonian H to be normalized so that $H = 0$ at the saddle point (and hence, on the separatrices). Then $H > 0$ in G_3 and $H < 0$ in $G_{1,2}$.

The averaged system in each of the domains G_i has the form

$$\dot{h} = \frac{\varepsilon}{T} \oint \left(\frac{\partial H}{\partial p} f_1^0 + \frac{\partial H}{\partial q} f_2^0 + \frac{\partial H}{\partial \lambda} f_3^0 \right) dt, \quad \dot{\lambda} = \frac{\varepsilon}{T} \oint f_3^0 dt, \tag{9}$$

where $T = \oint dt$ is the period of the unperturbed motion, $f_i^0 = f_i(p, q, \lambda, 0)$, $i = 1, 2, 3$, and the integrals are taken along the solution of the unperturbed system on which $H(p, q, \lambda) = h$. The averaged system for any of domains G_i can be extended to the separatrix by continuity by putting $\dot{h}|_{h=0} = 0$, $\dot{\lambda}|_{h=0} = f_{3C}^0$, where f_{3C}^0 is the value of function f_3^0 at the saddle point C. Now we can define the phase space of the averaged system. This is the object obtained by gluing along the surface $\{h = 0\}$ the phase spaces of the averaged systems for the domains G_i, $i = 1, 2, 3$ (Fig. 5).

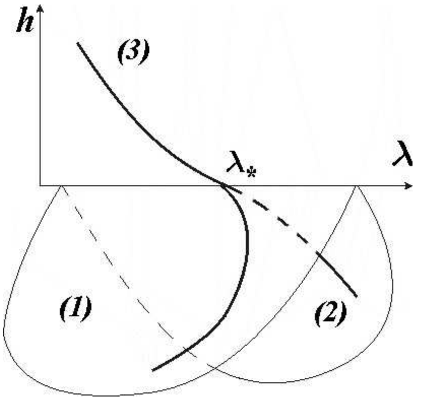

Figure 5. Phase space and trajectories of the averaged system in a case of separatrix crossings.

We define

$$\begin{aligned} \Theta_\nu(\lambda) &= -\oint_{l_\nu} \left(\frac{\partial H}{\partial p} f_1^0 + \frac{\partial H}{\partial q} f_2^0 + \frac{\partial H}{\partial \lambda} f_3^0 \right) dt, \quad \nu = 1, 2, \\ \Theta_3(\lambda) &= \Theta_1(\lambda) + \Theta_2(\lambda), \end{aligned} \tag{10}$$

where the integrals are taken along a separatrix in the unperturbed motion. These integrals are improper (since motion along a separatrix takes an infinite amount of time), but, it is not hard to verify that they converge for the normalization of the Hamiltonian that we have chosen. The quantities Θ_ν are assumed to be positive below. The value $-\varepsilon\Theta_\nu$ is close to the variation of the energy on the segment of perturbed trajectory close to an unperturbed separatrix l_ν. Therefore the positivity of Θ_ν ensures a sufficiently fast approach to a separatrix in the domain G_3 and a sufficiently fast moving away from a separatrix in the domains G_1 and G_2 for the majority of initial data.

The following assertions hold for the motion for $0 \le t \le 1/\varepsilon$ of a point with initial condition (p_0, q_0, λ_0), where $(p_0, q_0) \in G_3$ for $\lambda = \lambda_0$.

1 Denote by $(h_3(\varepsilon t), \lambda_3(\varepsilon t))$ the solution of the averaged system in the domain G_3 with initial condition $(H(p_0, q_0, \lambda_0), \lambda_0)$ (Fig. 5). Suppose that for some $\varepsilon t = \tau_* < 1$ this solution arrives to a separatrix: $h_3(\tau_*) = 0$. Then for $0 \le \varepsilon t \le \tau_*$ the behavior of H, λ along the true motion is described by the solution (h_3, λ_3) with accuracy $O(\varepsilon)$.

2 Let λ_* be the value of the parameter λ at the moment when the averaged solution arrives to the separatrix: $\lambda_* = \lambda_3(\tau_*)$. Then the probabilities of capture of the point $M_0 = (p_0, q_0, \lambda_0)$ in the domains G_1 and G_2 are calculated by the formulas:

$$P_i(M_0) = \frac{\Theta_i(\lambda_*)}{\Theta_1(\lambda_*) + \Theta_2(\lambda_*)}, \quad i = 1, 2. \tag{11}$$

3 Let $(h_1(\varepsilon t),\ \lambda_1(\varepsilon t))$ and $(h_2(\varepsilon t),\ \lambda_2(\varepsilon t))$ be solutions of the averaged system in the domains G_1 and G_2 with initial condition "on the separatrix", glued to (h_3, λ_3): $h_{1,2}(\tau_*) = 0$, $\lambda_{1,2}(\tau_*) = \lambda_*$ (see Fig. 5). Then for the majority of the initial points M_0 trapped in the domain G_ν the behavior of H, λ along the motion for $\tau_* \le \varepsilon t \le 1$ is described by the solution (h_ν, λ_ν) with accuracy $O(\varepsilon + \varepsilon|\ln\varepsilon|/(1 + |\ln|h_\nu(\varepsilon t)||))$.

4 The measure of the exceptional set of initial points whose motion cannot be described in this way, does not exceed $O(\varepsilon^r)$ for any prescribed $r \ge 1$.

Thus, to describe the motion we need to use the averaged system up to the separatrix, to compute the probability of being trapped in one domain or the other on a separatrix, and to use again the averaged system, starting at the separatrix in the domain in which the trapping has occurred. This scheme of analyzing the problem was first used in [32] for analysis of motion of charged quasiparticles. Detailed proofs of assertions 1–4 are contained in [39, 44].

Remark. The exposed approach for introducing concept of probability in the deterministic system under consideration can be interpreted as follows. Initial data are considered as random values with the smooth not depending on ε distrubution in a δ-neighborhood of a point M_0. In this case capture into a prescribed domain is the random event, and we can calculate its probability. The limit value of this probability when we proceed to the limit first as $\varepsilon \to 0$ and after that as $\delta \to 0$ is called the probability of capture of a point M_0 into the prescribed domain for the original deterministic system. There is another approach for introducing concept of probability [19, 67]. Let us add white noise with the variance $\varepsilon\delta$ to the right hand sides of equations (8) (one can consider also other random perturbations with strong enough mixing properties [19]). In this new system the capture into prescibed domain is truly a random event. The limit of the probability of this event as $\varepsilon \to 0$ and then as $\delta \to 0$ is by definition the probability of capture for the original deterministic system. For the classes of systems considered in [19, 67] this approach also leads to formulas (11) for probabilities.

Here is the list of some problems with separatrix crossings in the systems of the form (8):

- scattering of charged quasi-particles [32],
- tidal evolution of planetary rotation [21],
- motion of charged particles in the field of slowly evolving electrostatic wave [7],
- evolution of the orbital motions of satellites [62, 22, 38, 26],
- propagation of radio-waves in ionospheric wave-guides [23],
- tumbling of a rigid body under the action of small perturbations [51, 40],
- resonant heating of plasma in magnetic traps [63, 55],
- origin of a Kirkwood gap in the asteroid belt [66, 43],
- motion of charged particles in the Earth magneto-tail [13],
- rotation of dual-spin spacecraft [25, 24, 50].

For problems with separatrix crossings the small deviation of behavior of slow variables from the solutions of the averaged system also has property of sensitive dependence on initial data. Thus, this deviation should be treated as a random value. Properties of this deviation were studied mainly for the cases of Hamiltonian systems which depend slowly on time (Hamiltonian is $H = H(p, q, \varepsilon t)$) and of Hamiltonian systems with fast and slow motions (Hamiltonian is $H = H(p, q, y, x)$, pairs of canonically conjugated variables are (p, q) and $(y, \varepsilon^{-1}x)$. Equations of perturbed motion in these cases have the form (8) with slow variable $\lambda = \varepsilon t$ in the first case and $\lambda = (y, x)$ in the second case. Let a phase portrait of the unperturbed ($\lambda = \text{const}$) system be such as in Fig. 2. In every one of domains G_i we can introduce "action" variable $I(h, \lambda)$ of the unperturbed system as area surrounded by the line $H = h$ divided by 2π. "Action" is the first integral of the averaged system in every one of domains G_i (see, e.g., [1, 5]). In the approximation of the averaging method (which for problems under consideration is called an adiabatic approximation), "action" has a jump at separatrix crossing due to discontinuity of "action" at separatices. For example, if phase point from G_3 is captured into G_1 this jump of the"action" is equal to the area of G_2 divided by 2π. In exact system the "action" is an adiabatic invariant (approximate integral): for motion far from separatrices in every one of domains G_i value of the "action" is conserved with accuracy $O(\varepsilon)$ over time interval $O(1/\varepsilon)$. Adiabatic perturbation theory allows to construct improved adiabatic invariant J of the form $J = I + \varepsilon u(p, q, \lambda)$ such that for motion far from separatrices in every of domains G_i value of J is conserved with accuracy $O(\varepsilon^2)$ over time interval $O(1/\varepsilon)$ (see, e.g., [5]). It turned out that the improved adiabatic invariant J undergoes small deviation from prediction of the adiabatic approximation as a result of crossing by phase trajectory a narrow neighborhood of separatrices. The asymptotic formula for this jump of (improved) adiabatic invariant was obtained in pioneer work [65] for pendulum in the slowly varying gravity field, in [14, 41] for general case of Hamiltonian system depending on slowly varying parameters and in [42] for Hamiltonian system with slow and fast variables. Typical value of this jump is of order $\varepsilon \log \varepsilon$ for systems without special symmetry and of order ε for systems with some special symmetry (say, for passing from G_3 to G_1 or G_2 if $\Theta_1 = \Theta_2$, where Θ_ν is defined in Eq. (10) or for passing from rotation to oscillation and back for pendulum in the slowly varying gravity field). The formulas for jump of adiabatic invariant at separatrix express this jump through so called pseudo-phase which is a certain value characterizing a point where phase trajectory crosses separatrix. Pseudo-phase is sensitive to change of initial data and should be treated as a random value. Statistical distribution of pseudo-phase is uniform on $(0, 1)$ interval (for appropriate definition of probability similar to Eq. (7)). Thus, jump of adiabatic invariant at separatrix also should be treated as a random value; formulas expressing the jump through pseudo-phase provide statistical distribution of jumps. There are formulas expressing value of pseudo-phase through initial data for Hamiltonian system depending on slowly varying parameters [15] and for Hamiltonian system with slow and fast variables [59]. For different cases of separatix crossings near degeneration of a saddle asymptotic formulas for jump of adiabatic invariant at a separatrix are obtained in [17, 20]. An analog of asymptotic formulae for jumps of adiabatic invariant at separatrix crossings and variation of pseudo-phases in a class of dissipative systems is given in [10, 11].

Quasi-random deviations of exact motion from predictions of the averaging method and, in particular, jumps of adiabatic invariant at separatrix play an important role in problems with multiple separatrix crossings on a time interval of the length $\gg 1/\varepsilon$. Consider as an example motion of a pendulum in a slowly periodically varying gravity field. Phase portrait of the pendulum for fixed gravity field is shown in Fig. 6.

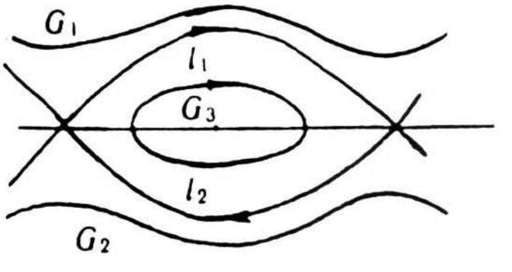

Figure 6. Phase portrait of a mathematical pendulum.

Separatrices divide it into the domain of oscillations and domains of rotations. For the domain of oscillations "action" on an unperturbed trajectory is the area surrounded by this trajectory divided by 2π. For domain of rotations "action" on an unperturbed trajectory is the area between this trajectory and axis of abscissa on any segment of this axis of the length 2π divided by 2π. There is a domain of initial data for which adiabatic approximations predicts that pendulum will periodically pass from regime of rotation to regime of oscillation and back. In this approximation, when pendulum passes from regime of rotation to regime of oscillation the value of the adiabatic invariant (i.e. value of "action") is doubled, and when pendulum passes back to regime of rotation the value of adiabatic invariant becomes two times smaller again. So, in this approximation the value of adiabatic invariant in the regime of rotation is the same for all times. However, every change of the regime leads to a small quasi-random jump of adiabatic invariant. Accumulation of these jumps for many separatrices crossings leads to quasi-random walk process. In many numerical experiments it was shown that this process leads to diffusion of adiabatic invariant for set of initial data of measure of order 1 (see, e.g. [12]). But there is no accurate theoretical prove of this. In Fig. 7a Poincaré section of the problem is presented.

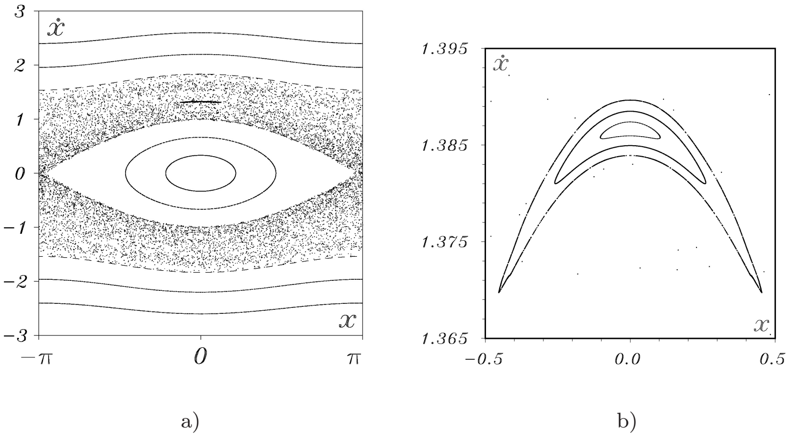

Figure 7. a) Poincaré section for a pendulum in a slowly periodically varying gravity field. b) Stability island.

The positions of several (eight) phase points at the time moments multiple to period of change of the gravity field are shown (calculations are performed by V.V. Sidorenko). By dotted lines we show boundary of the domain of separatrix crossings calculated in the adiabatic approximation. Smooth curves out of the domain of separatrix crossings are invariant curves of Poincaré return map. (Arnold theorem about perpetual adiabatic invariance [3] imply that the phase plane out of the domain of separatrix crossings up to a residue of a small measure is foliated by such curves.) Chaotically scattered points in the domain of separatrix crossings represent trajectory of one initial phase point. It turned out, that in this chaotic see there are islands of regular motion of total area of order 1 (this is the case for all problems where due to a special symmetry asymptotic formula for jump of the adiabatic invariant does not contain term of order $\varepsilon \log \varepsilon$). The islands surround stable fixed points of Poincaré return map. These fixed points correspond to stable periodic motions such that the pendulum passes from regime of rotation to regime of oscillation an back. There are $\sim 1/\varepsilon$ such periodic motions; stability island for each of them has an area at least $\sim \varepsilon$ [52, 53]. Total area of islands is therefore bounded from below by a value that does not depend on ε. (It is shown in [18] that area of individual stability island can not be bigger than a value $\sim \varepsilon$.) In Fig. 7a in the domain of separatrix crossings one can see the object which looks like a segment of a smooth curve. After enlargement of vertical scale one can see that this object is a smooth closed curve which is (approximately) a boundary of a stability island surrounding a fixed point of Poincaré map (Fig. 7b).

In problems without special symmetry, where jump of adiabatic invariant is of order $\varepsilon \log \varepsilon$, there are no stable periodic motions of period $\sim 1/\varepsilon$ deep inside the domain of separatrix crossings [52].

Probability phenomena associated with passages through separatrices exist also in volume-preserving systems close to integrable ones. In [6] the following volume-preserving system of equations is considered:

$$\dot{x} = -8xy + \varepsilon z, \quad \dot{y} = 11x^2 + 3y^2 + z^2 - 3, \quad \dot{z} = 2zy - \varepsilon x. \qquad (12)$$

This system describes a steady Stokes flow inside the unit sphere. For the unperturbed ($\varepsilon = 0$) flow all streamlines are closed curves except streamlines in the (invariant) meridional plane $x = 0$. This plane is a separatrix; it is filled up by streamlines passing from the point $x = 0$, $y = 1$, $z = 0$ to the point $x = 0$, $y = -1$, $z = 0$. It was shown numerically in [6] that the perturbed ($\varepsilon > 0$) flow possesses the following remarkable property: for arbitrarily small $\varepsilon > 0$ the entire unit sphere is the domain of streamline chaos, typical streamline under unbounded continuation tends to fill densely the entire unit sphere. Detailed study of probabilistic properties of this flow can be found in [54].

Another flow with similar property was considered in [64]. Under arbitrary small perturbation a large domain of streamline chaos arises inside the unit sphere.

These phenomena are associated with jumps of adiabatic invariant at separatrices. To study the effect of a perturbation we can use averaging method. For system (12) all trajectories of the averaged system are periodic. In the approximation of the averaging method all perturbed streamlines cross the separatrix of the unperturbed system. The flux of the vector field of the perturbation through a surface spanning an unperturbed streamline is an integral of the averaged system and adiabatic invariant of the exact perturbed system for motion far from the separatrix. When a perturbed streamline crosses a narrow neighborhood of the separatrix of the unperturbed system this adiabatic invariant undergoes a small jump which should be treated as random because of its sensitive dependence on initial data. Accumulations of these jumps in series of separatrix crossings lead to destruction of adiabatic invariance and streamline chaos. For the system in [64] the situation is similar, but the unit sphere can be divided into two parts: in one of these parts perturbed streamlines cross separatrix, and in the other part these trajectories do not cross separatrix. Streamline chaos arises in the part where perturbed streamlines cross separatrix. Asymptotic formulas for jump of adiabatic invariant in the above mentioned volume-preserving systems are obtained in [56, 57]. General formula of such kind is obtained in [58].

4. Passages through Resonances

Consider system with rotating phases (3) and corresponding averaged system (4). In the course of the evolution the frequencies of motion in Eq. (3) are changing slowly and at some time moments they become resonant, i.e. linearly rationally dependent. Due to the influence of resonances the actual motion can be considerably different from the one predicted by the averaging method. The two basic phenomena that are associated with the effect of a single resonance are capture into the resonance and scattering on the resonance.

Capture into a resonance can be described as follows. First the system evolves as it is predicted by the averaging method. At a certain time moment the system approaches a resonance. After that the system evolves in such a way that the resonant relation between frequencies is being kept approximately. As a result, after a time $\sim 1/\varepsilon$ the state of the system is completely different from the one predicted by the averaging method. Initial conditions for trajectories with a capture and trajectories without captures are mixed, if ε is small. Thus, it is reasonable to consider the capture as a random event and to calculate a probability of this event. Capture into a resonance was first discussed in [21, 34] in connection with problems of celestial mechanics.

Scattering on resonances is observed for trajectories that pass through resonances without being captured. Scattering is a deviation of such a trajectory from that predicted by the averaging method. During a passage through a narrow neighborhood of the resonance state slow variables undergo a small jump. The amplitude of this jump is very sensitive to the variation of initial conditions if ε is small. Thus, it is reasonable to consider this jump as random scattering on the resonance. Scattering on a resonance was first discussed in [16, 4].

Henceforth we will consider two-frequency systems: $m = 2$, $\varphi = (\varphi_1, \varphi_2)$, $\omega = (\omega_1, \omega_2)$. In two-frequency case in the frequency plane resonances are represented by straight lines with rational slopes passing through the origin of the coordinates. Such a simple structure of the resonant set simplifies the problem essentially. Under some additional assumptions it is possible to study an effect of each resonance separately of the others and combine the results together afterwards.

Study of the effect of a single resonance is based on reduction of the problem near the resonant surface to a standard "perturbed pendulum-like system" form. This transformation has been used in a number of works (see [5, 16, 30, 31, 33–37]). For the resonance $k_1\omega_1 + k_2\omega_2 = 0$ introduce resonant phase $\gamma = k_1\varphi_1 + k_2\varphi_2$, and normalized distance of point x from the resonant surface $\rho = (k_1\omega_1 + k_2\omega_2)/\sqrt{\varepsilon}$. Denote projection of the point x onto the resonant surface as σ. We introduce also "semi-slow" time $\theta = \sqrt{\varepsilon}t$ and use a prime to denote differentiation with respect to θ. One can show (see, e.g., [2, 5, 49]) that dynamics of variables γ, ρ, σ in a neighborhood of the resonant surface where $\rho = O(1)$ (called a resonant zone) is approximately described by the system of equations

$$\begin{aligned} \gamma' &= \rho + \sqrt{\varepsilon}\alpha_1(\gamma,\sigma), \\ \rho' &= L(\sigma) - \partial V(\gamma,\sigma)/\partial\gamma + \sqrt{\varepsilon}\rho\alpha_2(\gamma,\sigma), \\ \sigma' &= \sqrt{\varepsilon}\alpha_3(\gamma,\sigma). \end{aligned} \tag{13}$$

Here V, α_i are certain functions 2π-periodic in γ.

System (13) still contains a small parameter and can be treated by the perturbation theory. Putting in Eq. (13) $\varepsilon = 0$ we get for ρ, γ a Hamiltonian system with one degree of freedom; this system contains σ as a parameter. Hamiltonian of this system is

$$E(\rho,\gamma,\sigma) = \frac{1}{2}\rho^2 + V(\gamma,\sigma) - L(\sigma)\gamma, \quad \sigma = \text{const}. \tag{14}$$

Equation of motion can be written in the form

$$\gamma'' = -\partial V(\gamma,\sigma)/\partial\gamma + L(\sigma). \tag{15}$$

This system is called *a pendulum-like system* as it resembles a pendulum under the action of a constant torque (for a pendulum $V \sim -\cos\gamma$).

There are two basic types of the phase portraits of the Eq. (15): with an oscillation region, Fig. 8b, and without such a region, Fig. 8a.

In Fig. 8 the resonant surface is represented by the coordinate line $\gamma' = 0$. A resonant zone is a strip $|\gamma'| <$ const. Motion in oscillation region in Fig. 8b corresponds to motion of phase points that are captured into the resonance. Motion in supplementary region in Fig. 8b (the rotation region) corresponds to motion of phase points that cross the resonant zone without capture.

Now take into account terms $O(\sqrt{\varepsilon})$ in Eqs. (13). Under the action of these terms phase point may cross a separatrix in Fig. 8b and change the regime of the motion from a rotation to an oscillation or from an oscillation to a rotation. This means a capture into the resonance or an escape from the resonance respectively.

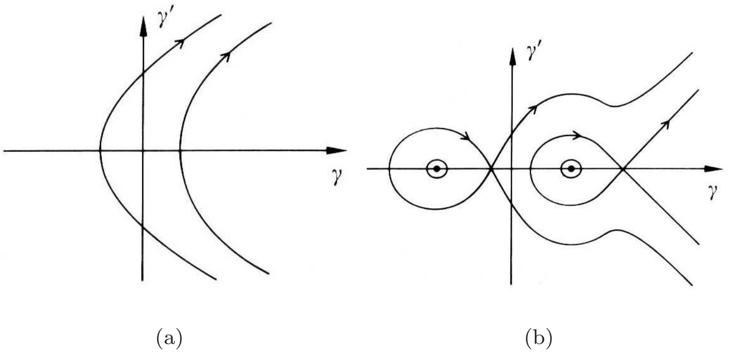

Figure 8. Phase portraits of a pendulum-like system: a) without an oscillation region, b) with an oscillation region.

Initial conditions for trajectories with a capture into a resonance and without a capture are mixed, if ε is small. Variation $\sim \varepsilon$ in initial value of x can change the motion considerably. It is reasonable to consider a capture as a random event and to calculate the probability of this event.

Consider motion of a phase point $(x(t), \varphi(t))$ that passes through the resonance without capture. Far from the resonant surface $x(t)$ oscillates with an amplitude $\ll \varepsilon^{1/2}$ near some solution of the averaged system. Before passage through the resonance this is a solution $X_-(t)$. After passage through the resonance this is a solution $X_+(t)$. Therefore, result of the passage through a narrow neighborhood of the resonant surface (actually, through a resonant zone) can be interpreted as a jump ΔX from one solution of the averaged system to another one. This jump creates scattering on resonance. Value of this jump depends on the value of the resonant phase γ at the moment of crossing of the resonant surface and is very sensitive to changes of initial data. Therefore, it is reasonable to consider the value of γ at the crossing of resonance as a random value. Then ΔX should be considered as a random value as well.

There are formulas for probability of capture into resonance, asymptotic description of motion of captured phase points, formulas for probabilistic distribution of amplitude of scattering on resonance (see, e.g. [49] and references therein). Versions of these formulas for Hamiltonian systems depending on slowly varying parameters and for Hamiltonian systems with slow and fast variables can be found in [47] and [48] respectively. Under rather general assumptions typical values of probabilities of captures and amplitudes of scattering are of order $O(\varepsilon^{1/2})$.

Examples of problems with passages through resonances and captures into resonances see, e.g., in [8, 21, 27–31, 34–36, 45, 61, 68, 69].

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