CRITICAL TIME FOR ACOUSTIC WAVES IN WEAKLY NONLINEAR POROELASTIC MATERIALS

Krzysztof Wilmanski

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, D 10117 Berlin, Germany

Summary The final time of existence (critical time) of acoustic waves is a characteristic feature of nonlinear hyperbolic models. We consider such a problem for poroelastic saturated materials whose material properties are described by Signorini-type constitutive relations for stresses in the skeleton and whose material parameters depend on the current porosity. In the one-dimensional case under considerations the governing set of equations describes changes of an extension of the skeleton, a mass density of the fluid, partial velocities of the skeleton and of the fluid, and a porosity. We rely on a second order approximation. Relations of the critical time to an initial porosity and to an initial amplitude are discussed.

MODEL

The problem of time of existence of weak discontinuity waves has been investigated for a two-component poroelastic model with a nonlinear constitutive relation for fluids [1]. It has been shown that such a model yields a finite time of existence of these waves and, consequently, a possibility of creation of shock waves if the fluid component is gaseous. In a series of subsequent papers (e.g. [2]) it has been shown by means of an asymptotic analysis that a dependence of material parameters of such a model on nonequilibrium changes of porosity yields for porosity soliton-like solutions of the Riemann problem. As changes of porosity were assumed to be given by a function of the fraction of partial mass densities ρ^F/ρ^S considerations were limited solely to a very rigid skeleton characteristic for rocks.

The present work is devoted to a one-dimensional wave analysis of a weakly nonlinear model of poroelastic materials in which material parameters depend on the current porosity. We limit the attention to granular materials for which Gassmann relations [3] can be applied. The model is based on three simplifying assumptions: 1) the relaxation of porosity is ignored, i.e. $\tau\to\infty$, where τ denotes the relaxation time in the porosity balance equation, 2) couplings between partial stresses are negligible, i.e. Q=0, where Q is the Biot's coupling parameter, and $\beta=0$, where β is the constant describing an explicit influence of nonequilibrium porosity changes on stresses, 3) the parameter c^S of the Signorini model is zero. Then partial stress tensors satisfy the following constitutive relations [4]

$$\mathbf{T}^S = \mathbf{T}_0^S + \left[\lambda^S(n)e + \tfrac{1}{2}\left(\lambda^S(n)+\mu^S(n)\right)e^2\right]\mathbf{1} + 2\left[\mu^S(n) - \left(\lambda^S(n)+\mu^S(n)\right)e\right]\mathbf{e}^S,$$

$$\mathbf{T}^F = -p^F\mathbf{1}, \quad p^F = p_0^F - \rho_0^F\kappa(n)\varepsilon, \quad e := tr\,\mathbf{e}^S, \quad \varepsilon := \frac{\rho_0^F - \rho^F}{\rho_0^F},$$

where $\mathbf{e}^S$ is the Almansi-Hamel deformation tensor of the skeleton, ρ^F is the current mass density of the fluid, $\rho^F{}_0$ – its constant initial mass density, n – porosity.

We assume deformations of the skeleton $\mathbf{e}^S$ and mass changes of the fluid ε to be sufficiently small for neglecting contributions of the order higher than two.

GOVERNING EQUATIONS

We consider a one-dimensional problem of plain deformations described by the fields

- e – deformation of the skeleton with $\mathbf{e}^S = e\,\mathbf{e}_x \otimes \mathbf{e}_x$, $\mathbf{e}_x$ - unit vector along x-axis,
- ρ^F – current mass density of the fluid component, or, equivalently, $\varepsilon = \left(\rho_0^F - \rho^F\right)/\rho_0^F$,
- v^S – velocity of the skeleton in the x-direction,
- v^F – velocity of the fluid in the x-direction,
- n – porosity.

Field equations for these fields result from the partial mass conservation laws, momentum balance equations, and the balance equation of porosity. Under assumptions presented in the first Section they have the form

$$\frac{\partial e}{\partial t} + v^S\frac{\partial e}{\partial x} - (1-2e)\frac{\partial v^S}{\partial x} = 0, \qquad \rho_0^S\sqrt{1-2e}\left(\frac{\partial v^S}{\partial t} + v^S\frac{\partial v^S}{\partial x}\right) - \frac{\partial}{\partial x}\left[E^S e - \tfrac{3}{2}\left(E^S - \mu^S\right)e^2\right] - n_E\pi_0\left(v^F - v^S\right) = 0,$$

$$\frac{\partial \varepsilon}{\partial t} + v^F\frac{\partial \varepsilon}{\partial x} - (1-\varepsilon)\frac{\partial v^F}{\partial x} = 0, \qquad \rho_0^F(1-\varepsilon)\left(\frac{\partial v^F}{\partial t} + v^F\frac{\partial v^F}{\partial x}\right) - \rho_0^F\frac{\partial}{\partial x}[\kappa\varepsilon] + n_E\pi_0\left(v^F - v^S\right) = 0,$$

$$\frac{\partial \Delta_n}{\partial t} + v^S\frac{\partial \Delta_n}{\partial x} + n_0\gamma\frac{\partial}{\partial x}\left(v^F - v^S\right) = 0, \qquad \Delta_n := n - n_E, \qquad n_E = n_0(1+\delta e),$$

where $\delta, \gamma, \pi_0 n_0$ are constants, $E^S := \lambda^S + 2\mu^S$, and a dependence of remaining material parameters on n is such that (compare [3]) $E_n^S := \partial E^S / \partial n < 0, \quad \mu_n^S := \partial \mu^S / \partial n < 0, \quad \kappa_n := \partial \kappa / \partial n > 0.$

CRITICAL TIME

Let us note that the solution of the porosity balance equation can be written in the form

$$n = n_0 [1 + \delta e + \gamma (e - \varepsilon)] + O\left(\min\left\{|e|^2, |\varepsilon|^2\right\}\right)$$

As we consider contributions up to the second order, material parameters depend solely on the linear contribution in the above relation. Simultaneously, it was shown in [3] that γ is at least one order of magnitude smaller than δ. Consequently, we have to account solely for the equilibrium porosity $n_E = n_0 (1 + \delta e)$ in relations for material parameters, i.e. E_n^S, μ_n^S, κ_n are constant and $E^S = E_0^S + E_n^S \delta e, \quad \mu^S = \mu_0^S + \mu_n^S \delta e, \quad \kappa = \kappa_0 + \kappa_n \delta e,$ where E_0^S, μ_0^S, κ_0 are constant. The balance equation of porosity separates from the remaining equations and it does not have any influence on critical times.

The time of existence (critical time) of acoustic waves as solutions for weak discontinuities of the above set of field equations is given, as usual, through the Bernoulli equation whose coefficients are determined by the solution of the corresponding eigenvalue problem. The derivation of the equation for evolution of amplitude is standard for hyperbolic systems (e.g. [6]). In our case the eigenvalue problem has the form

$$\frac{\partial u_A}{\partial t} + A^{AB} \frac{\partial u_B}{\partial x} = B^A, \quad i.e. \quad \left(A^{AB} - \lambda \delta^{AB}\right) r_B = 0, \qquad u_1 := e, \quad u_2 := \varepsilon, \quad u_3 := v^S, \quad u_4 := v^F,$$

where

$$\left(A^{1B}\right) = (u_3, 0, -(1 - 2u_1), 0), \quad \left(A^{2B}\right) = \left(0, \frac{1}{\rho_0^F} u_4, 0, 1 - u_2\right), \quad \left(A^{3B}\right) = \left(-\frac{E_0^S}{\rho_0^S} + \frac{2E_0^S - 3\mu_0^S - 2E_n^S \delta}{\rho_0^S} u_1, 0, u_3, 0\right),$$

$$\left(A^{4B}\right) = (\delta \kappa_n u_2, -\kappa_0 - \kappa_n \delta u_1 - \kappa_0 u_2, 0, u_4),$$

and λ denotes the eigenvalue corresponding to the right eigenvector r_B. The critical time follows from the equation for the amplitude Π

$$\frac{d\Pi}{dt} + \alpha_1 \Pi - \alpha_2 \Pi^2 = 0, \qquad \left[\left[\frac{\partial \mathbf{u}}{\partial x}\right]\right] = \Pi \mathbf{r},$$

where the double bracket denotes the difference of limits on the front of the wave and

$$\alpha_1 = l_A \frac{\partial B^A}{\partial u_B} r_B \frac{1}{\mathbf{r} \cdot \mathbf{l}}, \quad \alpha_2 = l_A \frac{\partial A^{AB}}{\partial u_C} r_B r_C \frac{1}{\mathbf{r} \cdot \mathbf{l}},$$

$$\left(B^A\right) = \left(0, 0, \frac{n_0 \pi_0}{\rho_0^S} [(1 + (1 + \delta) u_1)(u_3 - u_4)], -\frac{n_0 \pi_0}{\rho_0^F} [(1 + \delta u_1 - u_2)(u_3 - u_4)]\right),$$

provided a wave enters an undisturbed region. $\mathbf{l}$ denotes the left eigenvector.

We prove the existence of a finite critical time t_C for which $\Pi \to \infty$ in dependence on the initial porosity n_0. As an inverse problem, it indicates an existence of a critical porosity for a given time of existence of acoustic waves. This complements the analysis of critical values of porosity for flows in granular materials (the onset of fluidization, e.g. see: [7]) and of critical values of porosity in a static analysis of rocks (comp. [8]).

References

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