INSTABILITY THRESHOLDS OF FLOW BETWEEN EXACTLY COUNTER-ROTATING DISKS

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Summary We present the linear three-dimensional instabilities of the flow between exactly counter-rotating disks over the height-to-radius aspect ratio range $0.5 \leq \Gamma \leq 3$. The lowest Reynolds number threshold always corresponds to a non-axisymmetric and stationary eigenmode and the critical azimuthal wavenumber obeys $m_C \approx 2.2/\Gamma$. The axisymmetric instabilities are quite complicated, and are organized around various codimension-two points.

The von Kármán flow engendered by the differential rotation of the upper and lower bounding disks of a cylinder exhibits a large variety of phenomena, and depends on three parameters, which can be taken to be an angular velocity ratio $s \equiv \Omega_{up}/\Omega_{low}$, an aspect ratio $\Gamma \equiv H/R$, and a Reynolds number $Re \equiv \Omega_{low}R^2/\nu$. This flow is of growing interest to fluid dynamicists, but its three-dimensional patterns and transitions have as yet been explored for only a few parameter combinations, e.g. [1, 2, 3, 4, 5, 6, 7].

We focus on the case $s = -1$ in which the two disks rotate in equal and opposite directions. The system is then not only axisymmetric, but also invariant under simultaneous reflection in z and θ (or any horizontal axis), leading to the symmetry group $O(2)$. For low Re, the solution is unique and axisymmetric. For all values of Γ and Re, there exists an axisymmetric solution called the base state which is connected smoothly to the unique low-Re solution. We calculate the Reynolds numbers at which this solution becomes linearly unstable for $0.5 \leq \Gamma \leq 3.0$.

Newton's method, combined with a streamfunction-vorticity formulation, is used to calculate the base state. To calculate growth rates, we numerically integrate the Navier-Stokes equations linearized about the base state using a 3D code [6, 7] which combines a Fourier decomposition in θ with a staggered non-uniform grid in (r, z). For an axisymmetric base state, the stability computation separates into a family of decoupled subproblems each associated with an azimuthal wavenumber m. The leading eigenvalues are extracted as half the slope of the evolution of the logarithm of the energy corresponding to each m. The thresholds $Re_m(\Gamma)$ are calculated by interpolation and shown in figure 1.

The lowest threshold $Re_C(\Gamma) \equiv \min_m Re_m(\Gamma)$ is achieved for non-axisymmetric eigenmodes over the range $0.5 \leq \Gamma \leq 3.0$ and is shown in figure 3a. The fact that these modes are stationary is a consequence of the $O(2)$ symmetry of the configuration and of the basic flow. This leads to non-axisymmetric eigenmodes which are both stationary and invariant under simultaneous reflection in z and in θ; see figure 2. Figure 3b shows that the critical wavenumber increases approximately linearly with the radius, which is consistent with the idea that this is a generalized Kelvin-Helmholtz instability of an equatorial azimuthal shear layer occupying a constant proportion of the height.

Although the axisymmetric modes are not critical for these parameters, we present their thresholds in order to provide a complete framework which can be continued to other values of Γ or s. Figure 4a shows two pitchfork bifurcations existing for $\Gamma < 0.78$ and disappearing at an isola formation point. Figures 4b,c show two Hopf bifurcations with different frequencies co-existing for $\Gamma < 0.55$, where one of them disappears at a Takens-Bogdanov point. The rapid variation of the axisymmetric Hopf bifurcation threshold seen in figure 1 near $\Gamma = 1.6$ is due to an intersection of two complex eigenvalues. All the codimension-two points, both non-axisymmetric and axisymmetric, are shown in the table.

Competing wavenumbers or bifurcations	Γ	Re	Description
(4,3)	0.63	365	
(3,2)	0.95	310	
(2,1)	1.64	330	
H_1/P_2	0.51	2802	
H_1/H_2	0.535	2602	
H_2/P_2	0.55	2309	Takens-Bogdanov point
P_1/P_2	0.78	700	isola formation point
H_1	1.59	1860	intersection of two complex eigenvalues

References

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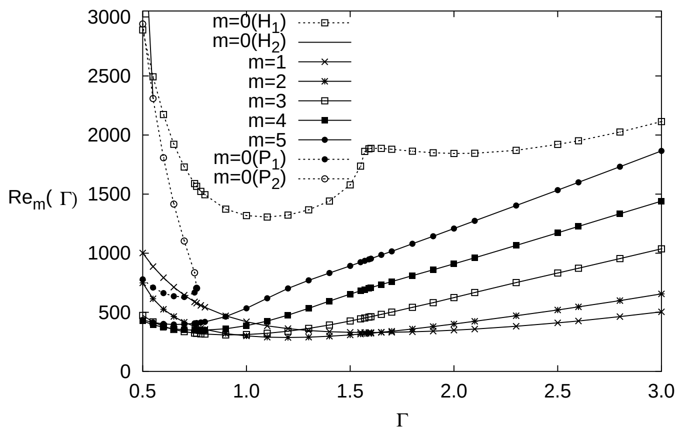


Figure 1. Thresholds $Re_m(\Gamma)$ for azimuthal modes $m = 0$ to 5 as functions of the aspect ratio Γ. The modes $m = 1$ to 5 are stationary while the $m = 0$ mode is stationary for $m = 0$ (P_1) and (P_2) and oscillatory for $m = 0$ (H_1) and (H_2).

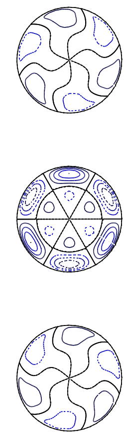
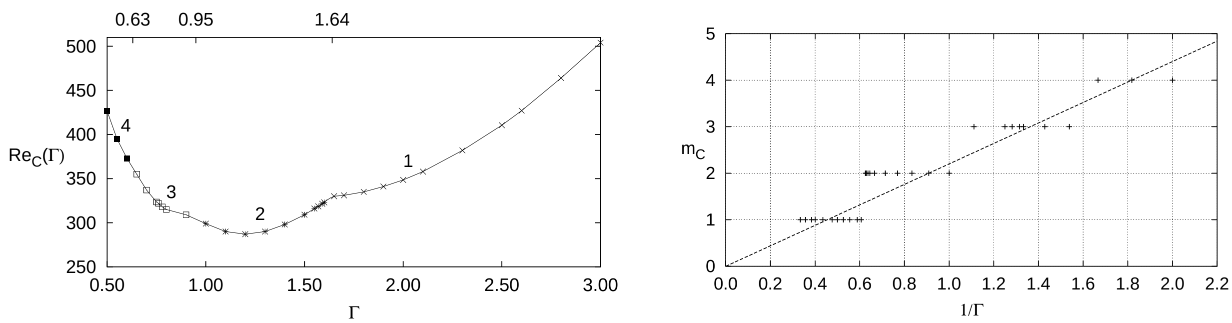
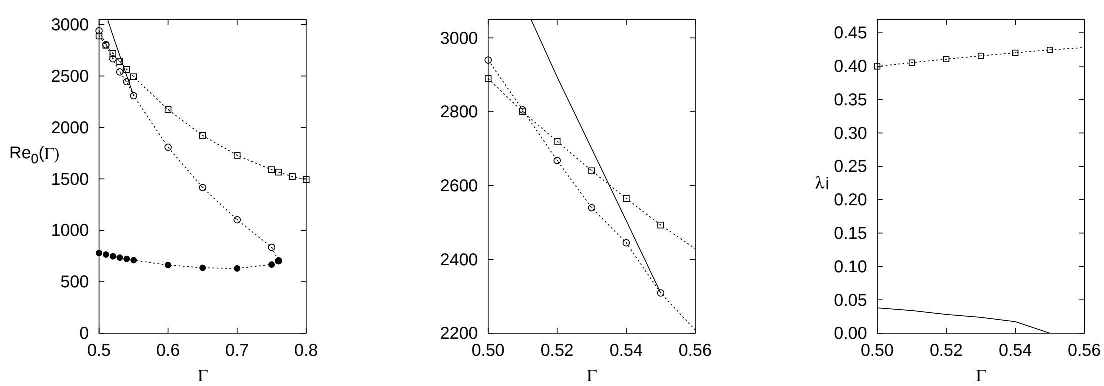


Figure 2. Vertical velocity contours of an $m = 3$ eigenvector at $z = \Gamma/4$, $z = 0$, and $z = -\Gamma/4$.



(a) (b)

Figure 3. (a) Critical Reynolds number $Re_C(\Gamma)$ as a function of Γ realised for different critical wavenumbers: $m = 1$ ($\times$), $m = 2$ ($*$), $m = 3$ ($\square$), $m = 4$ ($\blacksquare$). (b) Critical wavenumber m_C ($+$) as a function of $1/\Gamma$ with the indicative dashed line $m = 2.2/\Gamma$.



(a) (b) (c)

Figure 4. (a) Axisymmetric thresholds: two stationary bifurcations P_1 ($\bullet$) and P_2 ($\circ$) and two Hopf bifurcations H_1 ($\square$) and H_2 (—). P_1 and P_2 disappear at the isolation formation point at $\Gamma = 0.78$. (b) Enlargement of (a) showing the disappearance of H_2 at the Takens-Bogdanov point at $\Gamma = 0.55$. (c) Frequencies for H_1 ($\square$) and H_2 (—).