A BEM SOLUTION TO TRANSVERSE SHEAR LOADING OF BEAMS

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Summary In this paper a boundary element method is developed for the evaluation of the transverse shear stresses in beams of arbitrary simply or multiply connected constant cross section subjected in transverse shear loading. The transverse shear loading is applied at the shear center of the cross section, avoiding in this way the induction of a twisting moment. Two boundary value problems that take into account the effect of Poisson's ratio are formulated with respect to harmonic functions and solved employing a pure BEM approach. The evaluation of the transverse shear stresses is accomplished by direct differentiation of these harmonic functions, while both the coordinates of the shear center and the shear deformation coefficients are obtained from these functions using only boundary integration. Numerical examples with great practical interest are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method.

Statement of the problem

Consider a cylindrical beam of length L with a cross section of arbitrary shape, occupying the two dimensional multiply connected region Ω of the y,z plane bounded by the $K+1$ curves $\Gamma_1,\Gamma_2,\ldots,\Gamma_K,\Gamma_{K+1}$ as shown in Fig.1. Without loss of generality, it may be assumed that the beam end with centroid at point C is fixed, while the x –axis of the coordinate system is the line joining the centroids of the cross sections.

When the beam is subjected to torsionless bending arising from a concentrated load Q applied at the shear center M of its free end cross section, at a distance x from the fixed end the internal forces are the shear forces Q_y, Q_z being the components of the concentrated load Q along y and z axes, respectively and the bending moments M_y, M_z given as

$$M_y = -Q_z\left(L-x\right) \qquad M_z = Q_y\left(L-x\right) \tag{1}$$

Moreover, the normal component of stress acting on the beam cross section made from a homogeneous, isotropic and linearly elastic material is given as

$$\sigma_{xx} = -\left(\frac{M_y I_{yz} + M_z I_{yy}}{I_{yy} I_{zz} - I_{yz}^2}\right) y + \left(\frac{M_y I_{zz} + M_z I_{yz}}{I_{yy} I_{zz} - I_{yz}^2}\right) z \tag{2}$$

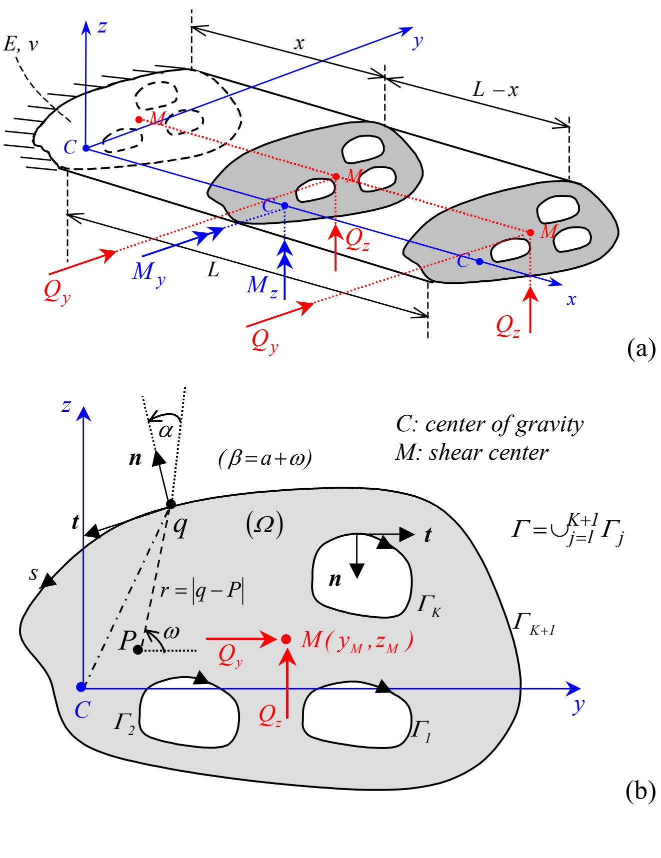


Fig.1. Prismatic bar subjected to torsionless bending (a) and two dimensional region Ω occupied by the cross section (b).

The evaluation of the shear stress distributions will be accomplished by regarding the beam subjected separately to Q_y and Q_z shear forces and obtaining the arising shear stress components by superposition.

Thus, considering the beam subjected to Q_z shear force and assuming a stress function $\Phi\left(y,z\right)$ having continuous partial derivatives up to the third order such that the compatibility conditions to be identically satisfied the transverse shear stress components τ_{xy}, τ_{xz} are expressed as

$$\tau_{xy} = \frac{Q_z}{B}\left[\frac{\partial \Phi}{\partial y} + \nu\left(I_{yz}\frac{y^2 - z^2}{2} - I_{zz} yz\right)\right] \qquad \tau_{xz} = \frac{Q_z}{B}\left[\frac{\partial \Phi}{\partial z} + \nu\left(I_{zz}\frac{y^2 - z^2}{2} + I_{yz} yz\right)\right] \tag{3a,b}$$

where it has been set $B = 2\left(1+\nu\right)\left(I_{yy} I_{zz} - I_{yz}^2\right)$. Substituting eqns.(3a,b) in the first equation of equilibrium and vanishing the traction vector normal to the boundary on the free surface of the beam, the partial differential equation governing the stress function $\Phi\left(y,z\right)$ and its boundary condition are obtained as

$$\nabla^2\Phi = 2\left(I_{yz}y - I_{zz}z\right) \quad \text{in } \Omega \qquad \frac{\partial\Phi}{\partial n} = \nu\left(I_{zz}yz - I_{yz}\frac{y^2-z^2}{2}\right)\cos\beta - \nu\left(I_{zz}\frac{y^2-z^2}{2} + I_{yz}yz\right)\sin\beta \quad \text{on } \Gamma \tag{4a,b}$$

while the coordinate y_M of the shear center M can be obtained from

$$y_M = \frac{1}{B}\left[\frac{\nu}{2}\int_\Omega\left(I_{zz}y + I_{yz}z\right)\left(y^2+z^2\right)d\Omega - \int_\Omega\left(z\frac{\partial\Phi}{\partial y} - y\frac{\partial\Phi}{\partial z}\right)d\Omega\right] \tag{5}$$

Similar relations with (3a,b), (4a,b) and (5) can be derived considering the beam subjected to Q_y shear force and defining as $\Theta(y,z)$ the corresponding stress function. Finally, the shear deformation coefficients a_y, a_z and a_{yz} using an energy approach are obtained from

$$a_y = \frac{A}{B^2}\int_\Omega\left[\left(\frac{\partial\Theta}{\partial y} - \nu\left(I_{yy}\frac{y^2-z^2}{2} - I_{yz}yz\right)\right)^2 + \left(\frac{\partial\Theta}{\partial z} - \nu\left(I_{yz}\frac{y^2-z^2}{2} + I_{yy}yz\right)\right)^2\right]d\Omega \tag{6a}$$

$$a_z = \frac{A}{B^2}\int_\Omega\left[\left(\frac{\partial\Phi}{\partial y} + \nu\left(I_{yz}\frac{y^2-z^2}{2} - I_{zz}yz\right)\right)^2 + \left(\frac{\partial\Phi}{\partial z} + \nu\left(I_{zz}\frac{y^2-z^2}{2} + I_{yz}yz\right)\right)^2\right]d\Omega \tag{6b}$$

$$\begin{aligned} a_{yz} = \frac{A}{B^2}\int_\Omega\Bigg[&\left(\frac{\partial\Theta}{\partial y} - \nu\left(I_{yy}\frac{y^2-z^2}{2} - I_{yz}yz\right)\right)\left(\frac{\partial\Phi}{\partial y} + \nu\left(I_{yz}\frac{y^2-z^2}{2} - I_{zz}yz\right)\right) + \\ &\left(\frac{\partial\Theta}{\partial z} - \nu\left(I_{yz}\frac{y^2-z^2}{2} + I_{yy}yz\right)\right)\left(\frac{\partial\Phi}{\partial z} + \nu\left(I_{zz}\frac{y^2-z^2}{2} + I_{yz}yz\right)\right)\Bigg]d\Omega \end{aligned} \tag{6c}$$

Integral Representations - Numerical Solution

The numerical solution of both boundary value problems concerning the stress functions $\Phi(y,z)$ (described by eqns.4a,b) and $\Theta(y,z)$ is similar. Thus, for the stress function $\Phi(y,z)$ employing the Green identity the following integral representation is obtained

$$\begin{aligned} \Phi(P) = &\frac{1}{4\pi}\int_\Gamma\left[\left(I_{yz}y - I_{zz}z\right)\left(2\ln r - 1\right)r\cos a - \left(I_{yz}\cos\beta - I_{zz}\sin\beta\right)\left(\ln r - 1\right)r^2\right]ds + \\ &\frac{1}{2\pi}\int_\Gamma\left[\Phi(q)\frac{\cos a}{r} - \left(\nu\left(I_{zz}yz - I_{yz}\frac{y^2-z^2}{2}\right)\cos\beta - \nu\left(I_{zz}\frac{y^2-z^2}{2} + I_{yz}yz\right)\sin\beta\right)\ln r\right]ds \quad P\in\Omega, q\in\Gamma \end{aligned} \tag{7}$$

The values of the function Φ inside the domain Ω are established from the integral representation (7) after evaluating the boundary values of Φ, which is achieved by writing eqn.(7) for the boundary points, using a special technique to overcome the singular line integrals and solving the resulting linear system of simultaneous algebraic equations.

Numerical examples

Numerical examples with great practical interest such as that presented in Fig.2 are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method.

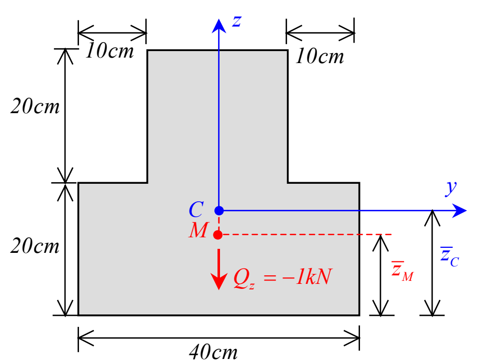
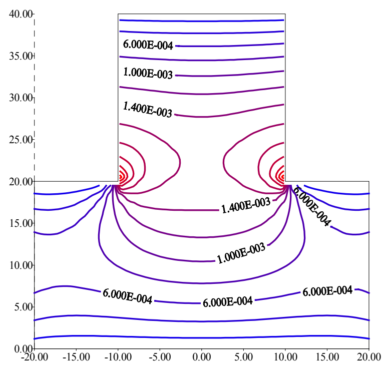
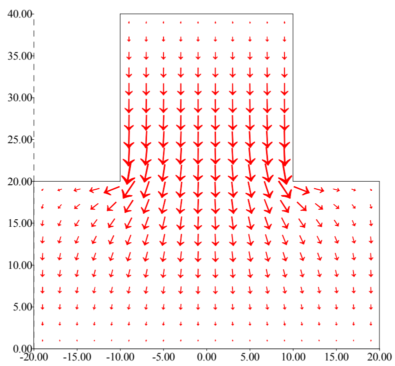


Fig.2. Geometry of the cross section.



Shear stresses τ_{xz} $(\nu = 0.20)$



Distribution of τ_{xz} $(\nu = 0.20)$

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