ON SHAPE OPTIMIZATION FOR EIGENVALUE PROBLEMS

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Summary For a broad class of static problems an optimality criterion of constant energy density along the designed boundary is proved earlier. In the present paper we prove a similar criterion for eigenvalue problems. This optimality criterion serve as the tool for more basic understanding and for idealized reference cases also as the basis for recursive procedures. Eigenfrequencies for in-plane vibrations as well as for out-of-plane vibrations of plates are optimized.

INTRODUCTION

In size optimization for minimum compliance with only a volume constraint, the necessary optimality criterion is uniform energy density. This criterion also holds for non-linear power-law, anisotropic elasticity, see [1] and [4].
For eigenfrequency optimization with size parameters a similar criterion specifies constant difference between amplitudes of elastic energy density and kinetic energy density, see [3] which describes the necessary modifications when Timoshenko beam theory is applied.
In shape optimization with prescribed size, say the thickness of a plate, we cannot expect constant energy density everywhere. For compliance minimization with only a single constraint on given volume the necessary optimality criterion will be constant energy density along the boundary to be designed, see [5] for extensions to non-linear power-law, anisotropic elasticity. and for the relations to strength optimization (stress constraints).
In the present paper we study shape optimization for eigenfrequency maximization and prove necessary optimality criteria, which expresses constant difference between amplitudes of elastic energy density and kinetic energy density along the boundary to be designed. The boundary may be internal (holes) as well as external. Numerical results for out-of-plane plate vibrations are recently published in [2]. In the present paper we concentrate on the theoretical aspects and also show results for in-plane plate vibrations.

THEORETICAL ASPECTS

A short version of the proof for the optimality criterion is here attempted. Let the shape of the boundary be described by localized design variables and choose two of these h_i and h_j. Each of these variables are related to areas A_i and A_j at the boundary, The differential dh_i implies the area differential dA_i and the differential dh_j implies the area differential dA_j. The constraint of unchanged total area (total volume) gives $dA_j = -dA_i$.
For a certain squared eigenfrequency ω^2, let the elastic energy amplitude related to the area A_i be U_i and let let the kinetic energy amplitude related to the area A_i be T_i. For the design variable h_j we have the corresponding quantities U_j and T_j. From general sensitivity analysis for eigenvalue problems we have, as a result of changing both design variables h_i and h_j, the differential

$$d(\omega^2) = (dU_i - dT_i)_{\text{fixed displacement mode}} + (dU_j - dT_j)_{\text{fixed displacement mode}} \tag{1}$$

assuming a normalization of displacement mode to give $T/\omega^2 = 1$. With fixed displacement mode we have (assuming homogeneous, explicit dependence on area) $dU_i = \frac{U_i}{A_i} dA_i$ and similar for dU_i, dT_i and dT_j. Inserting these relations in (1) and using the constraint condition $dA_j = -dA_i$ we get

$$d(\omega^2) = \left(\frac{U_i - T_i}{A_i} - \frac{U_j - T_j}{A_j} \right) dA_i \tag{2}$$

Then, a necessary condition for $d(\omega^2) = 0$ (stationarity of eigenvalue) for $dA_i \neq 0$ is

$$\frac{U_i - T_i}{A_i} = \frac{U_j - T_j}{A_j} \tag{3}$$

or in words: Constant difference between mean potential energy U_i/A_i and mean kinetic energy T_i/A_i for all design variables, i.e. with localized design variables: constant difference of energy densities along the boundary to be designed.

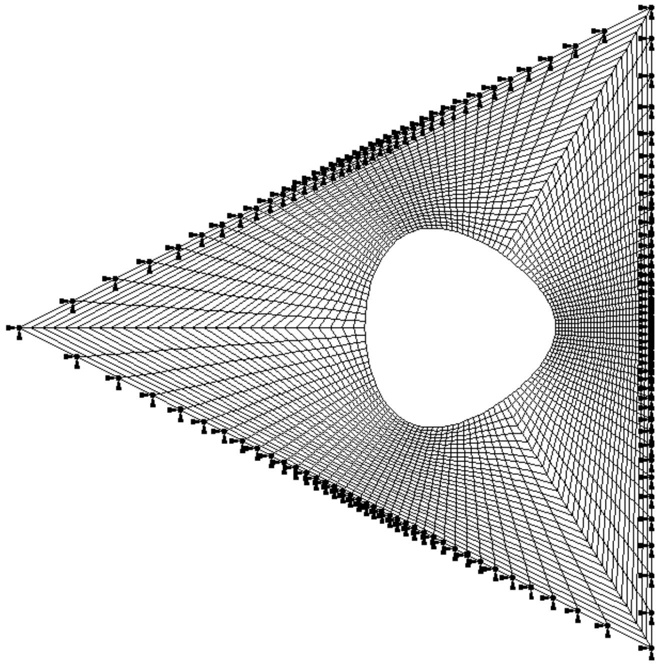

Figure 1. Finite element mesh of optimized plate showing the shape of the optimized hole for maximum first eigenfrequency of triangular plate.

NUMERICAL ASPECTS

Numerical solutions are based on the finite element modeling with extensive use of calculation on the element level. Multiplicity of eigenvalues are often encountered, especially when optimal design is attempted. Thus the method of analysis must be apple to deal with this. Therefore, the method of subspace iteration is applied.

Also the necessary sensitivity analysis must be apple to account for multiplicity. The solution is to locate the eigenmodes that also exist when the design is changed infinitesimal. This involves solving an eigenvalue-problem formulated in terms of mutual sensitivities, and the practical problem is merely to decide when multiplicity is present.

The numerical determination of optimal design parameters is iterative. For idealized problems simple recursive iterations can be based on the actual optimality criterion or a Newton-Raphson procedure can be setup to satisfy the criterion. With several constraints we used methods from mathematical programming.

NUMERICAL EXAMPLES

A number of numerical examples are presented in [2]. Figure 1 shows the optimal shape for a hole of given area in a triangular plate, where the frequency for the fundamental out-of plane free vibrations are maximized. For the present paper we also show a number of cases with in-plane vibrations, where part of the external boundary is subjected to design. Geometrical constraints, including the constraints from the chosen parametrization, can make it impossible to satisfy the optimality criterion completely, and we discuss this aspect in detail.

References

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