A MULTI-STEP TRANSVERSAL LINEARIZATION METHOD IN NONLINEAR DYNAMICS

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THE EXTENDED SUMMARY

An implicit family of semi-analytical integration methods, referred to as multi-step transversal linearization (MTL), is proposed for accurate, efficient and numerically stable integration of non-linear oscillators of interest in structural dynamics. The presently developed method is a multi-step extension and further generalization of the locally transversal linearization (LTL) method proposed earlier by the author (Roy 2001). The MTL-based linearization is achieved through a non-unique replacement of the nonlinear part of the vector field by a conditionally linear interpolating expansion of known accuracy, whose coefficients contain the discretized state variables defined at a set of grid points. For further illustration, consider the following system of *n* second order non-linear ordinary differential equations (ODE-s):

$$\{\ddot{X}\}+[C]\{\dot{X}\}+[K]\{X\}=\{Q(\{X\},\{\dot{X}\},t)\}+\{F(t)\} \qquad (1)$$

In the above equations, $[C]$ and $[K]$ are the damping and stiffness matrices respectively, $Q(X,\dot{X},t)=\{Q^{(j)}(X,\dot{X},t)\,|\,j=1,...,n\}$ is a non-linear vector function of the first two arguments, $\{F(t)\}$ is the externally applied force vector and $\{X\}$, $\{\dot{X}\}\in\Re^n$ are respectively the displacement and velocity vectors. Let the initial condition vector be denoted as $\{X(t_0)\}\underline{\underline{\Delta}}\{X_0\}$ and $\{\dot{X}(t_0)\}\underline{\underline{\Delta}}\{\dot{X}_0\}$. Now, let $I_1=[t_0,T_1\,|\,T_1>t_0)$ be a sub-interval of the time axis and let it be ordered into p smaller intervals as $t_0<t_1<\ldots<t_p=T_1$. The time step size $h_i=t_{i+1}-t_i=h$ is presently taken to be constant for convenience of further discussion. It is intended to derive the linearized ODE-s such that the linearized and non-linear vector fields remain identical at all the $(p+1)$ grid points, viz. $t_0,t_1,.......,t_p$. Let the non-linear and linearized flows, as parameterized by time t, be denoted by ϕ_t and $\overline{\phi}_t$ respectively. The non-linear flow, ϕ_t may be topologically viewed as a $C^k(k\geq 0)$ diffeomorphism on the associated (compact) manifold $M{:}\,\phi_t(\{X\}){:}\,M\times R\rightarrow M$ and $\phi_{t_0}\underline{\underline{\Delta}}\phi_0=id_M$, $\phi_{t_0}\,{}^0\phi_{h_1}\,{}^0\phi_{h_2}\,.........{}^0\phi_{h_p}=\phi_{t_p}$ provided that the vector field is autonomous (an *n*-dimensional non-autonomous field is equivalent to an *n*+1-dimensional autonomous field). The set of all such C^k diffeomorphisms $\phi_t(X)$, under the operation of composition '$\circ$' form a group, G_ϕ. Such compositions may be interpreted via *R*- and *Z*-group actions, defined as:

$$R\rightarrow G_\phi:t\rightarrow\phi_t,\quad Z\rightarrow G_\phi:j\rightarrow\phi_{t_j}\ \text{(R is the real line and Z is the set of integers)} \qquad (2)$$

Now, the MTL method attempts to derive a system of linearized ODE-s with solutions having identical Z-action for any t_j, $j\in Z$, and very similar *R*-action $\forall t\in R$ as the solutions of the nonlinear ODE-s. Towards this, the *j*-the component of the non-linear vector term $Q(X,\dot{X},t)$ is approximated as:

$$Q^{(j)}(X,\dot{X},t)\cong Q_M^{(j)}(t)=\sum_{k=0}^{p}Q^{(j)}(X_k,\dot{X}_k,t)\phi_k(t) \qquad (3)$$

where the basis set $\{\phi_k\,|\,k=0,...,p\}$, indexed on I_1, must satisfy $\phi_k(t_l)=\delta_{kl}$ (δ is the Kronecker delta), $\int_0^{ph}\phi_k(t)dt=1$ and $X_k=X(t_k),\dot{X}_k=\dot{X}(t_k)$. Interpolating Lagrange polynomials (ILP-s), distributed approximating functionals (DAF-s) (Wei *et al* 1998) and interpolating wavelets or interpolets (Saito and Beylkin 1993) are a few possible choices for $\{\phi_k\}$. An appropriate choice of the basis set may be dictated by the physics of the problem. Thus the MTL-based linearized form of equation (1) valid over I_1 is:

$$\{\ddot{Y}\}+[C]\{\dot{Y}\}+[K]\{Y\}=\{Q_M^{(j)}(X_k,\dot{X}_k,t\,|\,k=0,1,...,p)\}+\{F(t)\} \qquad (4)$$

where the first term on the RHS may be interpreted as a conditionally known forcing function. Since an exact solution of the above-linearized system is available, the formal accuracy of the MTL method as a function of the time step-size depends only the error of replacement of $Q^{(j)}(X,\dot{X},t)$ with $Q_M^{(j)}(t)$ as in equation (3). Now, by setting up the variational equations corresponding to equations (1) and (4), it may be shown that the tangent spaces of the non-linear and linearized systems are transversal almost everywhere in the associated phase space, and, in particular, at the grid points. Thus the discretized solution vector $\{X_k, \dot{X}_k \mid k = 1,...p\}$ may be interpreted as points of transversal intersections between the solutions of equations (1) and (4) at the grid points. Figure 1 provides a 1-dimensional schematic representation of the concept for $p = 4$. It may be shown that such intersections are indeed possible if the solution vector of equation (4) satisfies:

$$X_l = Y_l(X_k, \dot{X}_k),\ \ \dot{X}_l = \dot{Y}_l(X_k, \dot{X}_k),\ \ k = 0,1,..., p;\ \ l = 1,..., p \tag{5}$$

where $\{Y_l = Y(X_k, \dot{X}_k, t_l), \dot{Y}_l = \dot{Y}(X_k, \dot{X}_k, t_l) \mid k = 0,1,..., p\}$ denotes the linearized solution vector (which is an explicitly known function of $\{X_k, \dot{X}_k\}$) at $t = t_l$. Equation (5) constitutes a set of $2p$ algebraic equations in as many unknowns $\{X_l, \dot{X}_l \mid l = 1,..., p\}$ and may be solved using a Newton-Raphson or a non-linear iterative solver.

Presently, a limited numerical illustration of the method is provided for a few single- and two-degree-of-freedom nonlinear oscillators in their periodic and chaotic regimes. Figure 2 shows simulated periodic trajectories of a sinusoidally driven hardening Duffing oscillator (with a cubic non-linear term) obtained via MTL and Runge-Kutta (sixth order) schemes under different time step-sizes. The relatively superior numerical stability and convergence of the MTL method even under higher time-step sizes is amply clear. In particular, it is also observed that the Runge-Kutta method leads to overflows for $h = 0.21$. The MTL method used in this figure has been implemented via ILP-s with $p = 3$. It is interesting to note that a non-linear Hamiltonian system remains Hamiltonian under the MTL-based transformation. In fact, the non-uniqueness of the MTL method should be exploitable to tailor it to preserve any other invariants of motion (if they exist). Efforts are also underway to adapt the MTL principle for homogenization of ill-conditioned non-linear operators (e.g., dynamic wrinkling of membranes) of relevance in structural dynamics.

References:

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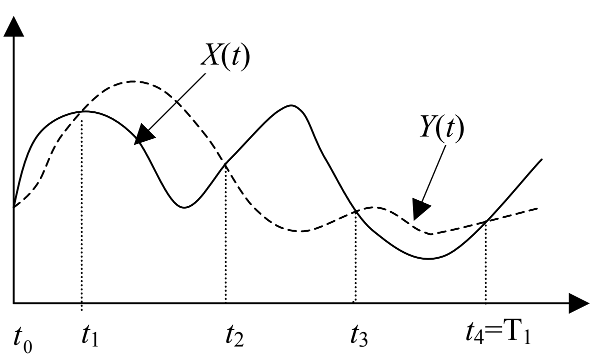

Fig. 1 A schematic representation of the MTL concept

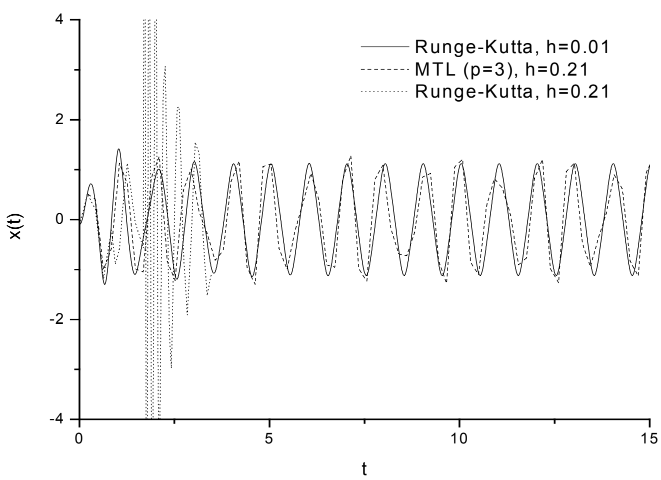

Fig. 2. Relative numerical stability of MTL vis-à-vis the sixth order Runge-Kutta method