# Two-layer stagnation point flows

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SUMMARY: Exact solutions of the Navier-Stokes equations are presented for orthogonal and oblique stagnation-point flow against a flat film resting on a plane wall.

We consider two-dimensional, orthogonal or oblique stagnation-point flow of a semi-infinite fluid with viscosity $\mu_1$ and density $\rho_1$, against a liquid film with viscosity $\mu_2$ and density $\rho_2$ resting on a plane wall, as depicted in figure 1. The interface between the two fluids is assumed to be and remain perfectly flat at all times. In the chosen system of coordinates the wall is located at $y = 0$, and the interface is located at $y = h(t)$, where $h(t)$ is the film thickness. The $x$ and $y$ velocity components are denoted by $u_j$, $v_j$ respectively, where $j = 1$ (2) refers to the lower (upper) fluid.

First, we consider the case of orthogonal stagnation-point flow, wherein the angle $\beta$ subtended between the incoming dividing streamline and the wall is $\beta = \pi/2$. Pozrikidis & Blyth (2004) showed that, in the limit of Stokes flow, the dividing streamline is straight, and the flow in the film is described by

$$u_1 = 2Gxy \quad v_1 = -Gy^2,$$

with corresponding pressure $p_1 = -(2\mu G + \rho_1 g)(y - h) + P_0(t)$, and the flow in the semi-infinite fluid is described by

$$u_2 = \frac{2G}{\lambda}\, x\,[y + h(\lambda - 1)], \qquad v_2 = -\frac{G}{\lambda}\,[y^2 + h(\lambda - 1)(2y - h)],$$

with corresponding pressure $p_2 = -(2\mu\, G + \rho_2 g)(y - h) + 4\mu h G(1 - \lambda) + P_0(t)$, where $G(t)$ is the strength of the stagnation-point flow, $P_0(t)$ is arbitrary, and $\lambda = \mu_2/\mu_1$ is the viscosity ratio. The Stokes-flow solution is valid when $Re \ll 1$, and breaks down sufficiently far from the origin. where $Re = rGh^2\rho/\mu$, with $r = (x^2 + y^2)^{1/2}$,

The solution at non-zero Reynolds numbers adopts a structure that is similar to that of the classical Hiemenz flow of a homogeneous fluid toward a plane wall. To begin, we write

$$u_j = x\,\frac{\partial f_j}{\partial y}, \qquad v_j = -f_j(y,t), \qquad p_j = x^2 P_j(t) + \phi_j(y,t), \tag{1}$$

where $P_j(t)$ are functions of time, $t$. Far from the wall, the flow matches the potential-flow solution $u = Gx$, $v = -Gy$, for constant $G$. Continuity of tangential and normal stresses must be satisfied at the interface, and the no-slip and no-penetration conditions are imposed at the wall. The kinematic condition at the interface yields an evolution equation for the film thickness $h(t)$, $\mathrm{d}h/\mathrm{d}t = -f_1(h)$. Substituting (1) in the Navier-Stokes equation and invoking the quasi-steady approximation to neglect the inertial time derivative on the left-hand side, we obtain a third-order ordinary differential

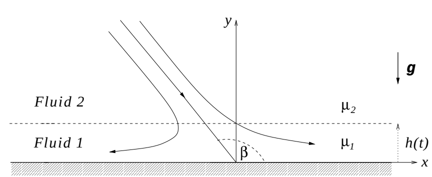


Figure 1: Planar two-fluid stagnation-point flow.

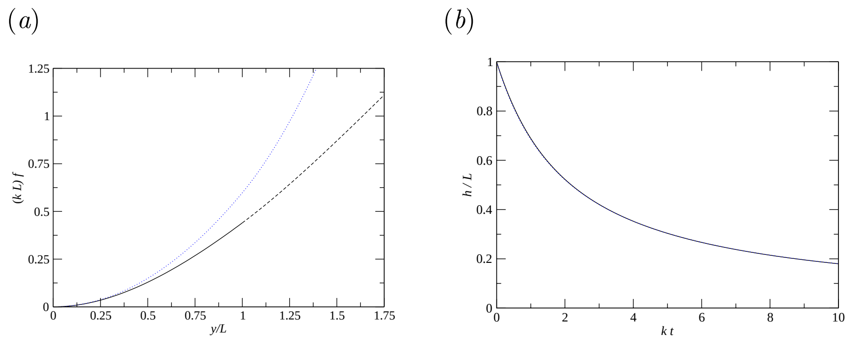


Figure 2: $\lambda = 0.5$, $h(0) = L = 1.0$: $(a)$ The stream function $f_j(y)$ at $kt = 0$. The solid line corresponds to fluid 1, and the broken line corresponds to fluid 2. The dotted line is the Stokes solution. $(b)$ $h(t)$ versus $kt$ for the quasi-steady (solid line) unsteady (dotten line) solutions.

equation to be solved numerically for $f_1(y)$. Alternatively, restoring the unsteady term, we obtain a third-order partial differential equation, which is to be solved numerically for $f_1(y,t)$. Either way, once $f_1$ is known, the evolution equation for the film thickness may be integrated forward in time using a standard numerical method. Sample results are shown in figure 1.

Next, we consider the case of oblique flow, wherein the incoming dividing streamline makes an angle $\beta$ with the wall, as shown in figure 2. For a homogeneous fluid, this problem was discussed by Stuart (1959) and Dorrepaal (1986). A solution valid for arbitrary Reynolds number may be sought by writing

$$u_1 = \frac{\partial g_1(y,t)}{\partial y} + x\,\frac{\partial f_1(y,t)}{\partial y}, \quad v_1 = -f_1(y,t), \quad p_1 = x^2 P_1 + x\pi_1(t) + \phi_1(y,t), \tag{2}$$

$$u_2 = \frac{\partial g_2(y,t)}{\partial y} + x\,\frac{\partial f_2(y,t)}{\partial y}, \quad v_2 = -f_2(y,t), \quad p_2 = x^2 P_2 + x\pi_2(t) + \phi_2(y,t), \tag{3}$$

where $g_j(y,t)$, $f_j(y,t)$, $P_j$, $\pi_j(t)$ and $\phi_j(y,t)$, $j = 1,2$ are to be found. Conditions similar to those for orthogonal flow must be satisfied at the wall and at the interface. The kinematic condition at the interface yields an evolution equation for the film thickness, which is numerically integrated in time.

## CONCLUSIONS

For both the orthogonal and oblique stagnation-point flow, exact solutions to the full Navier-Stokes equations are found in terms of ordinary and one-dimensional partial differential equations. The solutions are exact in the sense that no approximations are needed to simplify the full equations to their reduced form. Numerical simulations reveal that the quasi-steady approximation gives virtually indistinguishable results from the full unsteady calculations. Solutions of this type are also possible for axisymmetric flow.

### References

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