XXI ICTAM, 15–21 August 2004, Warsaw, Poland

ANALYTICAL SOLUTION FOR BENDING OF A CLAMPED ELLIPTICAL PLATE UNDER LATERAL LOAD AND IN-PLANE FORCE

Kenzo Sato*
*AKITA UNIVERSITY, Department of Engineerings and Information Science, Faculty of Education and Human Studies, Akita 010-8502, Japan

Summary On the basis of the ordinary thin plate theory, the analytical solution for the deflection by the bending of a clamped elliptical plate subjected to the uniform lateral load and in-plane force simultaneously is derived by the use of the elliptical cylinder coordinates. The solution is expressed in the form of an infinite series of Mathieu functions. The expressions for the bending moments are also derived analytically. Numerical calculation results are presented.

INTRODUCTION

Plane plates used as the elements of mechanical structures are commonly undergoing the action of various types of external forces in practical uses and therefore many studies on the bending of those plates have been reported during past years. About the bendings of rectangular plates and circular ones, which are subjected to the combined action of lateral load and in-plane force, the analytical solutions can be seen in the reference[1]. Also the studies on the bending of elliptical plates subjected to the action of external forces have been continued up to date[2]-[4]. An attempt to acquire the analytical solution for the bending of a clamped elliptical plate under the combined action of lateral load and in-plane force was made by Basuli[2]. However, his final solution is an approximation made by taking only a single Mathieu function of zero order[5]; besides, he gave no numerical information. Therefore, it cannot be regrettably stated that he had perfectly performed the theoretical analysis of the problem posed in his paper. The research on the purely analytical solutions for the bendings of plates contributes to the advance of the mathematical theory of elasticity; besides, the analytical solutions obtained successfully are very useful for confirming the validity of the other approximate numerical techniques. From the viewpoint of the usefulness of an elliptical plate as structural element and the importance of the research on the analytical solution for its bending, it is the purpose of this report to present the exact analytical solution for the bending of a clamped elliptical plate under the combined action of lateral load and in-plane force. Upon applying the orthogonality of Mathieu functions[6], the final deflection solution is expressed in the form of an infinite series of Mathieu functions. Moreover the expressions for bending moment distributions are also presented in this report. Some numerical calculation results are shown in a table and figures.

BASIC EQUATION AND ITS SOLUTION

The present problem is analyzed exactly in terms of the elliptical cylinder coordinates (ξ, η, z) which are related to the rectangular coordinates (x, y, z) by the relations $x = c\cosh\xi\cos\eta$, $y = c\sinh\xi\sin\eta$, $z = z$ with the semi focal length c. It is noted that c is denoted as $c = \sqrt{a^2 - b^2}$ in terms of the semi major axial-length a and the semi minor axial-length b. The differential equation for the deflection w of a clamped elliptical plate under uniform lateral load p and uniform in-plane compressive force T is given, by using the flexural rigidity D, as

$$\left\{\frac{2}{c^2(\cosh 2\xi - \cos 2\eta)}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\right\}\left\{\frac{2D}{c^2(\cosh 2\xi - \cos 2\eta)}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) + T\right\} w = p. \tag{1}$$

The boundary conditions of an elliptical plate clamped along its periphery ξ_o are

$$w = 0\,|_{\xi=\xi_o}, \quad \partial w/\partial\xi\,|_{\xi=\xi_o} = 0. \tag{2}$$

The solution w satisfying the boundary conditions (2) is, by using the Mathieu functions ce_{2m}, the modified Mathieu functions Ce_{2m} and the Fourier coefficients $A_{2u}^{(2m)}$ of Mathieu function[5],

$$\begin{aligned} w = \frac{pc^2}{8T}\sum_{m=0}^{\infty}\Big[&2A_0^{(2m)}\cosh 2\xi + A_2^{(2m)} - \frac{\mathrm{Ce}_{2m}(\xi,q)}{\mathrm{Ce}_{2m}(\xi_o,q)}\Big\{2A_0^{(2m)}\cosh 2\xi_o + A_2^{(2m)}\Big\} \\ &+ \sum_{u=0}^{\infty} f_{2u}(\xi_o,q)A_{2u}^{(2m)}\delta_u\Big\{\cosh 2u\xi - \frac{\mathrm{Ce}_{2m}(\xi,q)}{\mathrm{Ce}_{2m}(\xi_o,q)}\cosh 2u\xi_o\Big\}\Big]\mathrm{ce}_{2m}(\eta,q), \quad \delta_u = 2\,(u=0), 1\,(u\neq 0), \end{aligned} \tag{3}$$

where $q = Tc^2/4D$, ν denotes the Poisson's ratio and f_{2u} consist of the hyperbolic functions and the modified Mathieu functions. The bending moments M_ξ and M_η per unit length of sections of an elliptical plate perpendicular to ξ and η axes are, respectively,

$$\begin{aligned} M_\xi = -\big[pD/\{4T(\cosh 2\xi - \cos 2\eta)^2\}\big]\sum_{m=0}^{\infty}\big[&\{H_{2m}^{(3)}(\xi,q) - \nu H_{2m}^{(1)}(\xi,q)(a_{2m} - 2q\cos 2\eta)\}(\cosh 2\xi \\ &- \cos\eta)\mathrm{ce}_{2m}(\eta,q) - (1-\nu)H_{2m}^{(2)}(\xi,q)\sinh 2\xi\,\mathrm{ce}_{2m}(\eta,q) + (1-\nu)H_{2m}^{(1)}(\xi,q)\sin 2\eta\,\mathrm{ce}'_{2m}(\eta,q)\big] \end{aligned} \tag{4}$$

$$M_\eta = -\left[pD/\{4T(\cosh 2\xi - \cos 2\eta)^2\}\right] \sum_{m=0}^{\infty} \left[\{-H_{2m}^{(1)}(\xi,q)(a_{2m} - 2q\cos 2\eta) + \nu H_{2m}^{(3)}(\xi,q)\}(\cosh 2\xi - \cos\eta)\,\mathrm{ce}_{2m}(\eta,q) + (1-\nu)H_{2m}^{(2)}(\xi,q)\sinh 2\xi\,\mathrm{ce}_{2m}(\eta,q) - (1-\nu)H_{2m}^{(1)}(\xi,q)\sin 2\eta\,\mathrm{ce}'_{2m}(\eta,q)\right], \quad (5)$$

where $H_{2m}^{(1)}, H_{2m}^{(2)}$ and $H_{2m}^{(3)}$ consist of the hyperbolic functions and the modified Mathieu functions, and a_{2m} are the characteristic numbers of Mathieu functions[5].

NUMERICAL RESULTS

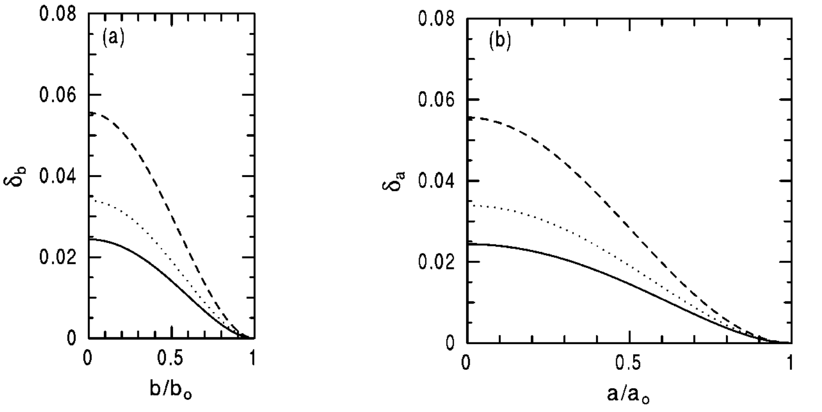
In numerical calculations for the elliptical plates having various aspect ratios a_o/b_o, the deflection and the moments were nondimensionalized as

$$\delta(\xi,\eta) = w(\xi,\eta)/(pb_o^4/D), \quad \alpha(\xi,\eta) = M_\xi(\xi,\eta)/(pb_o^2), \quad \beta(\xi,\eta) = M_\eta(\xi,\eta)/(pb_o^2). \quad (6)$$

It is noted that δ, α and β are directly proportional to w, M_ξ and M_η , respectively, and inversely proportional to p. Calculated results for various in-plane compressive force parameters λb_o^2 in the case of $a_o/b_o = 2$ are given in a table and figures. Poisson's ratio ν is 0.3. The numerical results are satisfactory as a whole and the detailed discussion will be presented at the conference.

Table 1. Convergence study of dimensionless deflection $\delta_c = \delta(0, \pi/2)$ at the plate center. $\lambda b_o^2 = 2$.

$m = u$	δ_c
0	0.028272
1	0.040965
2	0.042215
3	0.042126
4	0.042127
5	0.042127



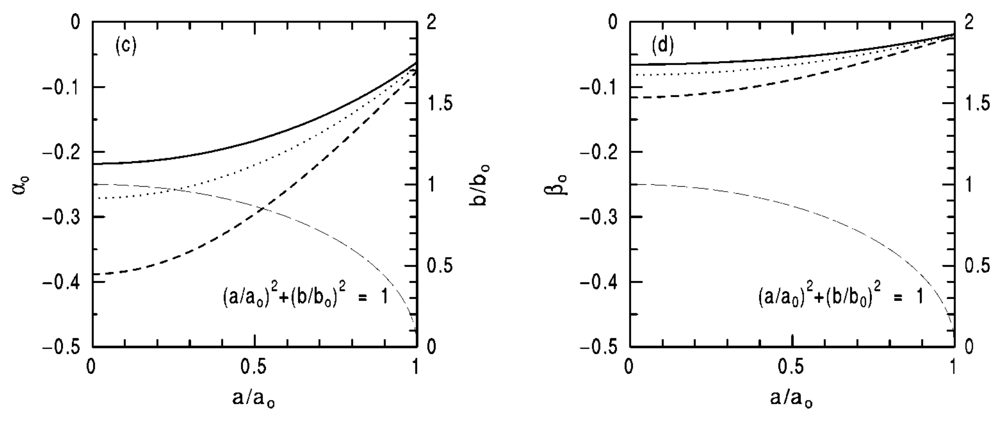


Figure 1. Dimensionless deflection δ and bending moments α, β : (a) $\delta_b = \delta(\xi, \pi/2)$ along the minor axis ; (b) $\delta_a = \delta(0,\eta)(a \le c), \delta(\xi, 0)(a \ge c)$ along the major axis ; (c) $\alpha_o = \alpha(\xi_o, \eta)$ along the periphery ; (d) $\beta_o = \beta(\xi_o, \eta)$ along the periphery. ——, $\lambda b_o^2 = -4$; ······, $\lambda b_o^2 = 0$; -----, $\lambda b_o^2 = 4$; ———, the periphery depicted by taking the dimensionless coordinate axes a/a_o and b/b_o scaled differently for representing the ellipticity 1/2.

References

[1] Timoshenko S. : Theory of Plates and Shells, 2nd ed., 311,378-395 McGraw-Hill, NY 1959.

[2] Basuli S. : Bending of Elliptical Plates under Simultaneous Action of Lateral Load and Force in the Middle Plane of the Plate. Bull. Calcutta Math. Soc. 54, 203-209, 1962.

[3] Vasilenko A.T. : Bending of an Anisotropic Elliptical Plate on an Elastic Plate on an Elastic Foundation. Int. Appl. Mech., 38-3 351-355, 1996.

[4] Laura P.A.A. and Rossit C.A. : Thermal Bending of Thin, Anisotropic, Clamped Elliptic Plates. Ocean Eng., 29, 485-488, 1999.

[5] McLachlan N.W. : Theory and Application of Mathieu Function, Dover, NY 1964.

[6] Sato K. : Vibration and Buckling of a Clamped Elliptical Plate on Elastic Foundation and under Uniform In-plane Force. Theor. and Appl. Mech., 51 49-62, 2001.