TRAVELLING WAVES IN A MODEL OF SKIN PATTERN FORMATION

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Summary. In this paper we study travelling wave solutions to a system of four non-linear partial differential equations, which arise in a tissue interaction model for skin morphogenesis. Under the "strong-body-force" assumption we prove the existence and uniqueness (up to a translation) of solutions with the dermis and epidermis cell densities being positive, which are a perturbation of a uniform epidermal cell density. We discuss the problem of the minimal wave-speed.

Travelling waves are not only a frequent phenomenon in the nature, but also they are of large cognitive value, since they are easy to be generated and measured in experiments. By measuring the profile and the speed of such a wave we can make estimations of the roles of various quantities characterising the medium under consideration.

The aim of this paper is a rigorous mathematical analysis of travelling wave solutions of a model proposed by Cruywagen and Murray [1] to study skin pattern formation.

According to the Cruywagen-Murray model, the skin consists of two layers, epidermis and dermis, separated by a thin basal lamina. The epidermis is modelled as a two-dimensional visco-elasic continuous medium. Under the biologically reasonable assumption that Reynolds number of the motion of the epidermis is low, the inertial terms of the equation of motion are ignored. The body force balances the elastic force, the viscous force, and the cell traction generated within the epidermis by a morphogen produced in the dermis. The force balance equation reads

$$\nabla \cdot \left\{ \frac{E}{1+\nu}\left[\boldsymbol{\varepsilon} - \beta_1 \nabla^2 \boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}\left(\theta - \beta_2 \nabla^2 \theta\right)\mathbf{I}\right] + \mu_1 \frac{\partial \boldsymbol{\varepsilon}}{\partial t} + \mu_2 \frac{\partial \theta}{\partial t}\mathbf{I} + \tau\, s\mathbf{I} \right\} = \rho \mathbf{u}\,, \tag{1}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ is the displacement at time t of a material point in the epidermis which was initially at position $\mathbf{x}$, $s(\mathbf{x},t)$ is the concentration of the signal chemical produced in the dermis, E is the constant Young modulus, ν is the constant Poisson ratio, μ_1, μ_2 are the constant shear and bulk viscosities, β_1, β_2 are positive constants, $\mathbf{I}$ is the unit 2×2 matrix, and τ is a positive parameter characterising the strength of the traction τs. The epidermis is attached to the basal lamina and ρ is a positive constant measuring the strength of this attachment. Next, $\boldsymbol{\varepsilon}$ is the strain tensor, and $\theta = \nabla \cdot \mathbf{u}$ is the dilatation.

This equation is supplemented by another two equations expressing the conservation laws of the epidermal cell density $N(\mathbf{x},t)$ and the dermal cell density $n(\mathbf{x},t)$. In the model of Cruywagen and Murray [1], the cells of the epidermis are mutually attached; so the only contribution to the cell density flux is convection. Therefore this equation is of the form

$$\frac{\partial N}{\partial t} = -\nabla \cdot \left(N \frac{\partial \mathbf{u}}{\partial t} \right). \tag{2}$$

The dermal cell density n changes due to random cell migration since the cells in the dermis are loosely packed, due to chemotaxis and due to mitosis, i. e. cell production. To model the random cell migration, the Fick's law of diffusion is used. The conservation equation of the dermal cell density $n(\boldsymbol{x},t)$ reads

$$\frac{\partial n}{\partial t} = \nabla \cdot \left(d(n,N)\nabla n\right) - \alpha \nabla \cdot \left(n \nabla e\right) + m(n), \tag{3}$$

where d is the coefficient of diffusion, which can be a function of the cell densities N and n, α is chemotaxis coefficient, $e(\mathbf{x},t)$ is the concentration of the signal chemical produced in the epidermis, and $m(n)$ describes the mitosis.

The aim of this paper is a rigorous mathematical analysis of travelling wave solutions of the above equations, under a simplifying assumption that the force exerted by the basal lamina is much larger than the other forces acting on the epithelium, i.e. the dominant part of Eq. (1) is the right hand side. Under this assumption, in the lowest order of approximation the displacement $\mathbf{u} = \mathbf{0}$, and the problem reduces to a study of travelling wave solutions to the system (2) and (3). Using the existing results concerning the latter degenerated problem we the existence of travelling waves with positive dermis cell density to the full system (1) – (3). The mathematical apparatus is the Implicit Function Theorem in a suitable Banach space. Also a discussion of the minimal wave-speed problem is carried out..

Reference

1. Cruywagen G. C , Murray J. D. On a tissue interaction model for skin pattern formation. *Journal of Nonlinear Science*. 1992; **2**; 217-240.