Equivalent Stochastic Linearization as an Alternative to Solving the Fokker-Planck Equation

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Statistical, or equivalent stochastic, linearization was originally introduced [1, 2, 3] as an *approximate* procedure for estimating the mean and variance of the stationary response of a nonlinear dynamic system. The procedure has been widely applied to a variety of problems [4], but for the sake of simplicity of exposition, the discussion here is limited to an oscillating mass m with linear damping and odd, nonlinear, restoring force $g(x)$, excited by a stationary zero-mean random force $f(t)$, described by the equation

$$m\ddot{x} + c\dot{x} + g(x) = f(t) \tag{1}$$

For this special case the displacement respone has zero mean, and the statistic of interest is just the mean square $E[x^2]$. The basic idea of stochastic linearization is to replace the nonlinear restoring force $g(x)$ by an equivalent linear force kx. One criterion for selecting k, proposed by all three of the pioneers [1,2,3], is that the expectation of the mean square of the equation deficiency should be a minimum; *i.e.*,

$$\frac{d}{dk}E[\{kx - g(x)\}^2] = 0 \tag{2}$$

According to the pioneers, the expectations in (2) should ideally be evaluated on the basis of the true probability distribution of the nonlinear response. Since the actual distribution of the nonlinear response was then unknown, the pioneers made the additional approximation of assuming a Gaussian distribution in their illustrative examples. Caughey [3] did however note that if the correct nonlinear distribution could have been employed, then the linearization procedure would have produced the correct response statistics for the case where $f(t)$ was Gaussian white noise. Subsequently most authors have adopted the Gaussian assumption in their applications of equivalent linearization, There has, however, been some confusion as to whether the Gaussian distribution involved is an approximation for the true nonlinear distribution, as originally proposed, or is the particular Gaussian distribution which describes the response of the equivalent linear system to Gaussian white noise. In [5] it is shown that these two assumptions lead to two distint procedures, but that the second procedure is more labor intensive and generally less accurate than the first.

A few authors have investigated stochastic linearization based on non-Gaussian probability distributions. In [6] the linearization procedure is re-interpreted as a technique for selecting the optimum distribution out of a one-parameter family of possible probability distributions. The novel contribution of the present paper is the recognition that the Maxwell-Boltzmann equation provides a one-parameter family of distributions which includes the the true probability distribution of the stationary solution to the nonlinear equation (1) when the excitation is Gaussian white noise. In the classical approach [7], the Fokker-Planck equation is solved to select the true distribution. Here it is shown that the same result is obtained by applying dimensional analysis and equivalent stochastic linearization. This means that for any odd restoring force $g(x)$ for which the potential energy

$$G(x) = \int_0^x g(\xi) d\xi \tag{3}$$

is known, the true probability density function for the white-noise response is

$$p(x) = \frac{\exp\{-G(x)/G_0\}}{\int_{-\infty}^{\infty} \exp\{-G(x)/G_0\} dx} \tag{4}$$

where the magnitude of the reference energy G_0 is readily fixed by equivalent stochastic linearization. In this application the linearization procedure is a tool for obtaining a *true* solution. The procedure is illustrated for an array of particular examples, including power-law oscillators, Duffing's system, the double-well oscillator, and oscillators with trancendental restoring forces.

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