RAYLEIGH-LIKE SURFACE WAVES ON A NONLINEAR LAYERED ELASTIC HALF SPACE

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Summary Nonlinear modulation of Rayleigh-like Surface Waves on a layered half space is examined by employing an asymptotic perturbation method. It is shown that the first order slowly varying amplitude of the wave modulation is governed by a nonlinear Schrödinger (NLS) equation. Then the effects of material nonlinearities on the existence of bright (envelope) and dark (hole) surface solitons are discussed for both hypothetical and real material models.

FORMULATION OF THE PROBLEM

Recently, the propagation of nonlinear Rayleigh-like surface waves on a layered elastic half space has been the subject of several investigations [1,2,3]. In these works mainly the propagation of long nonlinear waves are considered and Benjamin-Ono (BO)-like evaluation equations are derived to describe the wave field asymptotically. Then the existence of nonlinear periodic and solitary wave solutions are discussed via BO-like equations. These examinations are restricted effectively to the small wave number, low frequency region since the layer is assumed to be thin and in some works also is assumed to be linear [2.3]. In this work, the problem is investigated for a finite nonlinear layer.
In a rectangular frame (X,Y,Z), it is assumed that the layer occupies the region between the planes Y=h and Y=0 and the half space lies in the region Y<0, where h>0 represents the thickness of the layer. Then the Rayleigh waves are supposed to propagate along the positive X-axis and their displacements lie in the (X,Y)-plane, the saggital plane. Thus the equation

$$x = X + u_1^{(\nu)}(X,Y,t)\ ,\ \ y = Y + u_2^{(\nu)}(X,Y,t)\ ,\ \ z = Z \tag{1}$$

represent the wave motion, where $u_1^{(\nu)}$ and $u_2^{(\nu)}$ are the displacement components in X and Y directions, (x, y, z) are the spatial rectangular coordinates and t is the time. Here and henceforth superscript value ν=1 refers to the layer and ν=2 to the half space. The constituent materials are assumed to be homogenous, isotropic and compressible elastic, and the stress constitutive relations are taken in the following quadratic nonlinear form;

$$\begin{aligned} T_{Kk}= &\big[\lambda(tr\widetilde{\mathbf{E}})+\tfrac{\lambda}{2}u_{p,M}u_{p,M}+\tfrac{1}{2}(6l+3m+n)(tr\widetilde{\mathbf{E}})^2-\tfrac{1}{2}(m+n)(tr\widetilde{\mathbf{E}}^2)\big]\delta_{Kk}+\big[2\mu-(m+n)\\ &(tr\widetilde{\mathbf{E}})\big]\widetilde{E}_{KL}\delta_{Lk}+\big[\lambda(tr\widetilde{\mathbf{E}})\big]u_{k,K}+2\mu\widetilde{E}_{KL}u_{k,L}+\mu u_{p,K}u_{p,L}\delta_{Lk}+n\widetilde{E}_{KN}\widetilde{E}_{NL}\delta_{Lk} \end{aligned} \tag{2}$$

where T_{Kk} is the first Piola-Kirchoff stress tensor, E_{KL} is the linear Lagrangian deformation tensor and λ, μ are the Làme and l,m,n are Murnaghan constants. In the absence of body forces, the equations of motion in the reference state are

$$T^{(\nu)}_{\beta 1,\beta} = \rho_0\ddot{u}_1^{(\nu)}\ ,\ T^{(\nu)}_{\beta 2,\beta} = \rho_0\ddot{u}_2^{(\nu)} \quad \nu = 1,2 \tag{3}$$

where the subscripts preceded by comma indicate partial differentiation with respect to X or Y and an over-dot represents the partial differentiation with respect to t. The free boundary Y=h of the layered half space is assumed to be traction free and stresses and displacements are continuous at the interface Y=0 and the displacements tend to zero as the depth increases. Hence the following boundary conditions are written;

$$\begin{aligned} &T_{21}^{(1)}=0\ ,\ T_{22}^{(1)}=0 \quad \text{on} \quad Y=h \quad \text{and} \quad u_1^{(2)},u_2^{(2)} \to 0 \quad \text{as}\ Y\to-\infty \\ &u_1^{(1)}=u_1^{(2)},\ \ u_2^{(1)}=u_2^{(2)} \quad \text{and} \quad T_{21}^{(1)}=T_{21}^{(2)},\ \ T_{22}^{(1)}=T_{22}^{(2)} \quad \text{on}\ \ Y=0\ . \end{aligned} \tag{4}$$

THE NONLINEAR MODULATION OF RAYLEIGH-LIKE WAVES

To investigate how the slowly varying amplitude of a weakly nonlinear Rayleigh-like wave is modulated by nonlinear self interaction, the method of multiple scales is employed. For this purpose, the new independent variables

$$x_i = \epsilon^i X\ ,\ \ y_i = \epsilon^i Y\ ,\ \ t_i = \epsilon^i t \quad i = 0,1,2, \tag{5}$$

are introduced where $\epsilon > 0$ is a small parameter representing the weakness of the nonlinearity. $\{x_0, y_0, t_0\}$ are fast variables describing the fast variations in the problem while $\{x_1, x_2, y_1, y_2, t_1, t_2\}$ are slow variables to describe the slow variation. The displacements $u_1^{(\nu)}$ and $u_2^{(\nu)}$ are assumed to be functions of these new variables and they are expanded in the following asymptotic power series in ϵ [4];

$$u_1^{(\nu)}=\sum_{n=1}^{\infty}\epsilon^n u_{1n}^{(\nu)}\ ,\ \ u_2^{(\nu)}=\sum_{n=1}^{\infty}\epsilon^n u_{2n}^{(\nu)}. \tag{6}$$

Then following the usual procedure of an asymptotic analysis a hierarchy of problems to determine $u_{1n}^{(\nu)}$ and $u_{2n}^{(\nu)}$ are obtained. These problems, at each step, are linear and the first order problem is simply the classical linear wave problem in a layered linear elastic half space [5]. This first order problem is solved by assuming that the phase velocity c of the waves satisfies the inequality $c_{1_T} < c_{1_L} < c < c_{2_T} < c_{2_L}$ where c_{ν_L} and c_{ν_T} denote linear longitudinal and shear wave velocities in the layered half space. Under these assumptions and since also the harmonic resonance phenomena is excluded in the analysis the solutions are found as

$$\begin{aligned} u_{11}^{(1)} &= A_1\left[R_1 e^{ikp_L y_0} + R_2 e^{-ikp_L y_0} - p_T R_3 e^{ikp_T y_0} + p_T R_4 e^{-ikp_T y_0}\right] e^{i\phi} + c.c. \\ u_{21}^{(1)} &= A_1\left[p_L R_1 e^{ikp_L y_0} - p_L R_2 e^{-ikp_L y_0} + R_3 e^{ikp_T y_0} + R_4 e^{-ikp_T y_0}\right] e^{i\phi} + c.c. \\ u_{11}^{(2)} &= A_1\left[R_5 e^{kv_L y_0} + iv_T R_6 e^{kv_T y_0}\right] e^{i\phi} + c.c. \\ u_{21}^{(2)} &= A_1\left[-iv_L R_5 e^{kv_L y_0} + R_6 e^{kv_T y_0}\right] e^{i\phi} + c.c. \end{aligned} \tag{7}$$

where a "$c.c.$" symbol denotes the complex conjugate of the preceding terms and $\mathcal{A}_1$ is the complex function of the slow variables $\{x_1, x_2, t_1, t_2\}$ representing the first-order slowly varying amplitude of the wave modulation. Also, $\phi = kx_0 - \omega t_0$, $p_\alpha = (c^2/c_{1_\alpha}^2 - 1)^{1/2}$, $v_\alpha = (1 - c^2/c_{2_\alpha}^2)^{1/2}$, $(\alpha = L, T)$, and R_i's are some constants. Obviously, to complete the first order solutions $\mathcal{A}_1$ has to be determined. This can be achieved by examining the higher order perturbation problems. A compatibility condition in the second order perturbation problem related with the fundamental mode yields that

$$\mathcal{A}_1 = \mathcal{A}_1(x_1 - V_g t_1, x_2, t_2) \tag{8}$$

i.e., $\mathcal{A}_1$ remains constant in a frame of reference moving with the group velocity V_g of the waves. And then a compatibility condition in the third order problem related with the fundamental mode yields the following equation for $\mathcal{A} = k\mathcal{A}_1$;

$$i\frac{\partial \mathcal{A}}{\partial \tau} + \Gamma \frac{\partial^2 \mathcal{A}}{\partial \xi^2} + \Delta |\mathcal{A}|^2 \mathcal{A} = 0\,. \tag{9}$$

where $\tau = \omega t_2$, $\xi = k(x_1 - V_g t_1)$, $\Gamma = \frac{k^2}{2\omega}\frac{d^2\omega}{dk^2}$ and the coefficient Δ depends on nonlinear material parameters. The equation (9) is a NLS equation and it is derived in various branches of science and engineering to describe the nonlinear self modulation of waves asymptotically.

CONCLUSIONS

It is known that the stability of the solutions of the NLS equation describing the asymptotic wave field and the existence of various types of soliton solutions depend on the sign of the product of the coefficients of the dispersion term, Γ, and the coefficient of the nonlinear term, Δ. Therefore to investigate the dependence of the sign of $\Gamma\Delta$ on the nonlinearity of the constituent materials, the numerical evaluation of this product with respect to the wave number is performed by fixing the linear material constants. The behavior of Γ and Δ are also analyzed in the limit as $h \to 0$ and for fixed k, i.e. in the thin layer limit. It has been observed that, in this limit the behavior of Δ is dominated by the nonlinear properties of the half space, but as h increases (or kh grows) the effects of the nonlinear properties of the layer on Δ begin to increase. It has been also observed that, depending on the linear properties of the layered half space, in the neighbourhood of the critical wave number k_c satisfying the second harmonic resonance condition, $|\Delta|$ grows without bound, i.e., $|\Delta|$ goes to infinity as $k \to k_c$. Also, in the thin layer limit $|\Delta|$ grows without bound for the layered half-space models whose half space made of a nonlinear material. But, if the half space is linear then Δ goes to zero in this limit. These results also indicate that in the thin layer limit, the nonlinear properties of the half space dominates the nonlinear wave modulation.

References

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