STATICAL MODELS TO ILLUSTRATE SPECIAL INSTABILITIES

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Summary The paper shows models to illustrate some special instability. Three models belong to the cusp catastrophe. They have point-like instability, stable-X or unstable-X point of bifurcation, respectively. Two models illustrate classes of the double cusp catastrophes, where the homogeneous quartic parts of the potential energy functions have two distinct or two coincident real roots. The equilibrium paths of the perfect and some imperfect structures and several imperfection-sensitivity curves will be presented.

Classification of instabilities

Elastic structures carrying one-parameter conservative loads will be analyzed. Following the primary equilibrium path the structure loses its stability when its total potential energy function has a degenerate critical point. Catastrophe theory classifies the degenerate critical points ([1]). Thom's theorem has given canonical forms of the typical catastrophes when the number of the parameters is less than six. The potential energy functions depend on the load parameter and the imperfections. The load parameter has a special role which is different from the role of the imperfection parameters, so they cannot be mixed by the necessary transformations to achieve the canonical forms. This is why a subclassification is needed for the stability theory of structures. Changing the load parameter we will have a route in the parameter space. These routes may cross the bifurcation set qualitatively different ways ([2]), i.e. they belong to different classes according to the stability theory.
In the case of the (standard and dual) cusp catastrophe we have the stable-symmetric and unstable-symmetric point of bifurcations, respectively, when the route cross the bifurcation set, and typically there is a cut-off point when the route remains outside of the cusped region. But special cases of these routes result in lip singularity ([2], p. 85) where the equilibrium path has a point-like instability. If the routes remain inside of the cusped region we have a beak-to-beak singularity which will be called as a stable-X or an unstable-X bifurcation in the case of the standard and dual cusp, respectively.
If the smallest critical value of the load parameter is a double eigenvalue of the Hessian matrix of the potential energy function, and the function is symmetric in both buckling modes, then the perfect system must exhibit on of the double cusp catastrophes, which are not included in Thom's theorem (because they need more than five control parameters). Double cusp catastrophes are classified according to the root structure of the homogeneous quartic parts of the (potential energy) functions ([3]). In the case of the Augusti model ([2], p. 144) there are four real roots (and four secondary paths cross the primary equilibrium path). If a guyed cantilever was supported by more than four springs ([4]) then there were no real roots, and the number of the secondary paths was equal to the number of the springs. Here we show models to which two real roots belong, and they are distinct or coincident.

Model for the point-like instability

The model shown in Figure a) consists of two linear, two rotational springs and two telescopic elements. In the case of the perfect structure both linear springs have unit length and their stiffness are c=1. The stiffness of the rotational springs are 3, and they are free of stress when the telescopic members are horizontal. The vertical dead load acts on the middle hinge. We introduce two imperfections: the length of the linear springs are $l = 1 + \varepsilon_1$, and there is also a horizontal load with magnitude ε_2. The critical load of the perfect structure is $\Lambda^{cr} = 0$. The potential energy function can be split into an active and a passive part. The active part has the form:

$$V_a = 0{,}5x^4 + \left(0.125\Lambda^2 - \varepsilon_1\right)x^2 - \varepsilon_2 x \,,$$

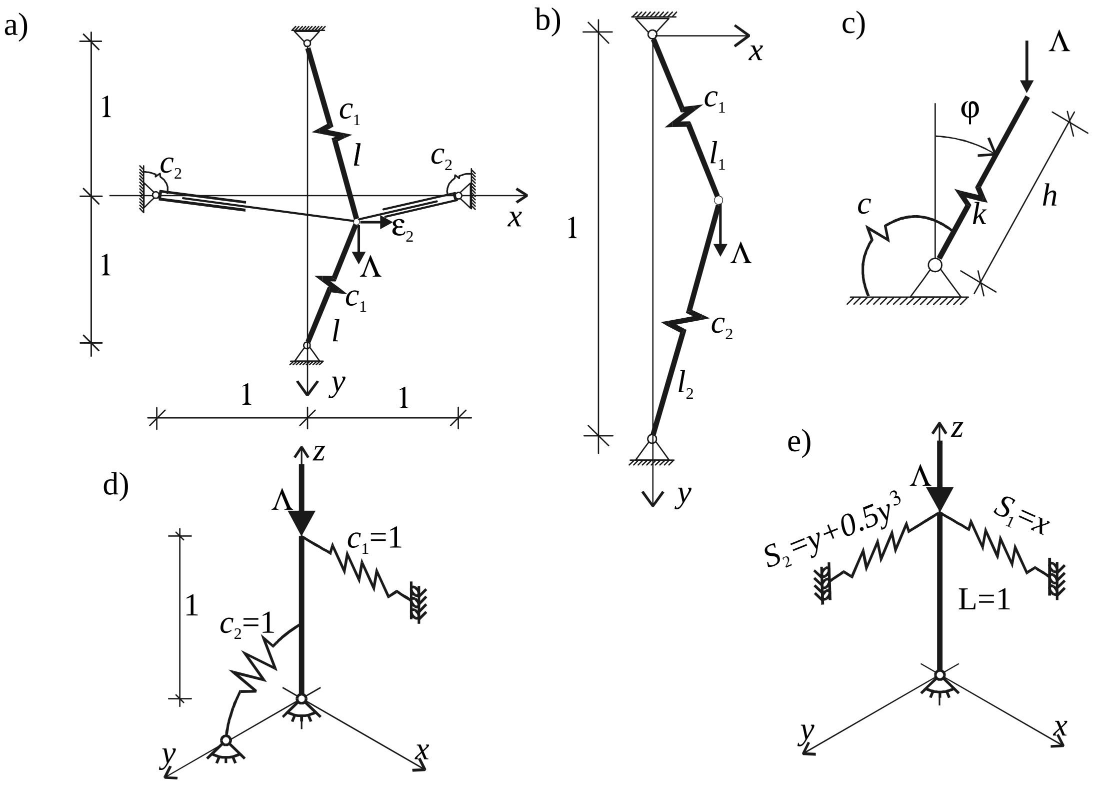
so equilibrium path of the perfect structure is a straight line and every point of it belongs to stable states. The structure is not sensitive to imperfections. For some imperfections the equilibrium path separates into a stable and a close curve.

Model for the stable-X point of bifurcation

The model (Figure b) consists of two linear springs. c_i denotes the stiffness in both tension and compression, and l_i is the stress free length of the ith spring. Let $c_1 = 1.985177808, c_2 = 1, l_1 = 0.4, l_2 = 0.7$. In the critical state: $x^{cr} = 0$, $y^{cr} = 0.5157563683, \Lambda^{cr} = 0.4455533422$. The active part of the potential energy can be transformed to the form:

$$V_a = 1{,}494072598x^4 - 0.6706428261\lambda^2 \,,$$

where $\lambda = \Lambda - \Lambda^{cr}$. There are three straight equilibrium paths: the vertical one is unstable while the other two are stable. The structure is not sensitive to imperfections, imperfect structures do not lose their stability when the load increases.



Model for the unstable-X point of bifurcation

The model is a (vertical) hinged cantilever (Figure c) comprising a link with normal rigidity k=4, pinned to a rigid foundation and supported by a linear rotational spring of stiffness c=1. In the critical state: $\varphi^{cr}=0$, $h^{cr}=0.5$, $\Lambda^{cr}=2$. The active part of the potential energy can be transformed to the form:

$$V_a=-\frac{1}{12}\varphi^4+\frac{1}{8}\lambda^2\varphi^2-\varepsilon\varphi\,,$$

where $\lambda=\Lambda-\Lambda^{cr}$ and $\varphi=\varepsilon$ for the unloaded imperfect structure. There are three straight equilibrium paths: the vertical one is stable while the other two are unstable. The structure is very sensitive to imperfections: $\lambda^{cr}=\pm(12\varepsilon)^{1/3}$.

Models for double cusp catastrophes

The model (shown in Figure d) comprising a rigid link, pinned to a rigid foundation and supported by two springs. The linear spring is always parallel with axis x. The trivial equilibrium path is given by $x=y=0$. The critical load is $\Lambda^{cr}=1$. The truncated Taylor expansion of the energy function is:

$$V=-\frac{1}{8}\left(x^4+2x^2y^2-\frac{1}{3}y^4\right)-\lambda\left(\frac{x^2+y^2}{2}+\frac{x^4+2x^2y^2+y^4}{8}\right).$$

There are two secondary paths: one of them belongs to positive values of λ, and is unstable, the other belongs to negative values of λ, and its points are critical. These equilibrium states falls into three paths for a region of the imperfections.

The model (shown in Figure e) consists of two springs, they are always parallel with a coordinate axis. The force in the second spring depends nonlinearly on the elongation. The critical load is $\Lambda^{cr}=1$. The truncated Taylor expansion of the energy function is:

$$V=-\frac{1}{16}y^6-\frac{1}{8}x^4-\frac{1}{4}x^2y^2-\frac{\lambda}{2}\left(x^2+y^2\right).$$

There are two equilibrium paths, both belongs to negative values of λ. One of them is unstable, the points of the other are critical. These equilibrium states fall into three paths for a region of the imperfections.

References

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