TRANSIENT GROWTH IN DEVELOPING PLANE AND HAGEN POISEUILLE FLOW

Peter W. Duck
Department of Mathematics, University of Manchester, Manchester, England

Summary The problem of the stability of developing entry flow in both two-dimensional channels and circular pipes is investigated. The basic flow is generated by uniform flow entering a channel/pipe, which then provokes the growth of boundary layers on the walls, until (far downstream) fully developed (Poiseuille) flow is attained; the length for this development to be $\mathcal{O}$(Reynolds number) $\times$ the channel/pipe width/diameter. This enables the use of high-Reynolds-number theory, leading to boundary-layer-type equations; as such there is no need to impose heuristic parallel-flow approximations. The resulting flow is shown to be susceptible to significant, three-dimensional transient (initially algebraic) growth in the streamwise direction, and as such large amplifications to flow disturbances are shown to occur (followed by ultimate decay far downstream). It is suggested that this initial amplification of disturbances is a possible mechanism for flow transition, with steady disturbances being the most 'dangerous'.

INTRODUCTION AND FORMALISM

The stability of Hagen-Poiseuille flow has intrigued scientists for more than a century since Reynolds' experimental investigations. It is reasonably well accepted that *fully developed* Hagen-Poiseuille flow is linearly stable and yet, in practice, most pipe flows are turbulent. In a carefully controlled experimental study of this flow the following sequence of events is usually observed. Below a Reynolds number, Re of $\approx 2,000$ any disturbance introduced into the flow will decay and the flow can be said to be stable in practical terms. Above this value it is possible to maintain laminar flow and the highest recorded Re for laminar flow is $\approx 100,000$ which was achieved with extraordinary amounts of care.

In this paper we consider the stability of developing flow inside two-dimensional channels and circular pipes. At the inlet, the flow is taken to be uniform, and then boundary layers form on the channel/pipe walls, which downstream eventually merge to form fully developed flow. The lengthscale for fully developed flow to become established is very long at large Reynolds numbers - $\mathcal{O}(Re)\times$ width/diameter of the channel/pipe. Close to the inlet the boundary layers resemble flat-plate (Blasius) boundary layers, which are susceptible to streamwise algebraic growth of three-dimensional disturbances ([1], [2]). In this paper it is shown how this class of disturbance is important in the flow development problem.

Consider the two-dimensional channel case (the pipe case develops in an analogous manner). We take coordinates $L(xRe, y, z)$, origin at the leading edge of the lower wall, which lies along $y = 0$, with the flow directed along the positive x direction; z is the crossflow direction. Here L denotes the semi-width of the channel, and the Reynolds number $Re = \frac{U_\infty L}{\nu}$, with U_∞ the incoming (uniform) flow velocity and ν the kinematic viscosity of the fluid (assumed to be incompressible). Throughout it is assumed $Re \gg 1$, enabling the use of the boundary-layer equations to be entirely rational. The flow velocity is then written in the form $\mathbf{u} = U_\infty(U, Re^{-1}V, Re^{-1}W)$, whilst the pressure develops in the form $p = \rho U_\infty^2(P_0(x) + Re^{-1}P_1(x) + Re^{-2}P_2(x, y, z) + \ldots)$, ($\rho$ being the density of the fluid). Taking the leading-order (in Re) terms in the Navier-Stokes equations leads to

$$U_x + V_y + W_z = 0 \quad , \quad U_t + UU_x + VU_y + WU_z = -P_{0x} + U_{yy} + U_{zz},$$
$$V_t + UV_x + VV_y + WV_z = -P_{2y} + V_{yy} + V_{zz} \quad , \quad W_t + UW_x + VW_y + WW_z = -P_{2z} + W_{yy} + W_{zz}, \qquad (1)$$

where $(\frac{L}{U_\infty})t$ denotes dimensional time. Note the y and z components of the momentum equation involve the *third* term in the pressure expansion.

We now decompose the velocity field (U, V, W) into a base flow and a small amplitude ($\mathcal{O}(\delta)$) perturbation, as follows

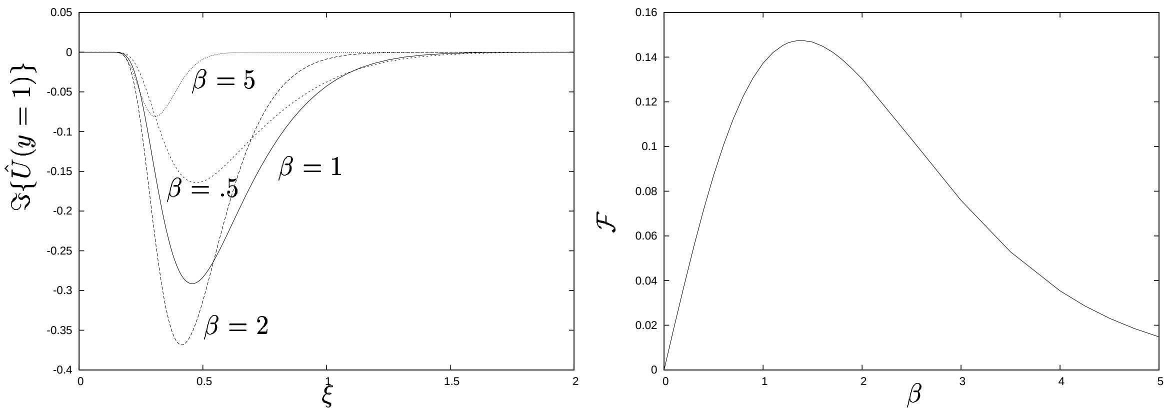
$$(U, V, W, P_2) = (U_0, V_0, 0, 0) + \delta(\hat{U}, \hat{V}, \hat{W}, \hat{P}_2)e^{i\omega t + i\beta z} + \mathcal{O}(\delta^2), \qquad (2)$$

ω being the frequency of the disturbance and β the crossflow wavenumber.

In order to generate accurate results for the disturbance field, it was found necessary to treat both the base flow and the disturbance field with great care, employing a double numerical grid which mimicked the analytic behaviour as $x \to 0$. It is possible to deduce from the analysis that as $x \to 0$, the disturbance field admits streamwise-growing algebraic behaviour, corresponding to that found in [1], [2], [3] and [4], provided the disturbances are three-dimensional (i.e. $\beta \neq 0$). Eigenfunctions corresponding to these modes then provided the initial conditions for the disturbance field, which was then extended downstream using routine parabolic marching techniques (a similar procedure was adopted for the base flow).

RESULTS

Results for the streamwise perturbation velocity along the line of symmetry within the channel are shown in 1(a) for four spanwise wavenumbers (where $\xi = \sqrt{x}$); these results are all for the case of steady disturbances ($\omega = 0$) since it is found that these are generally the most dangerous, i.e. provoke the largest flow response. Of particular note is the value of β that generates the most significant response. This can obviously be measured in a variety of means, but here we define a



(a) Streamwise perturbation velocity along symmetry line

(b) Variation of response function with β

Figure 1. Channel results ($\omega = 0$)

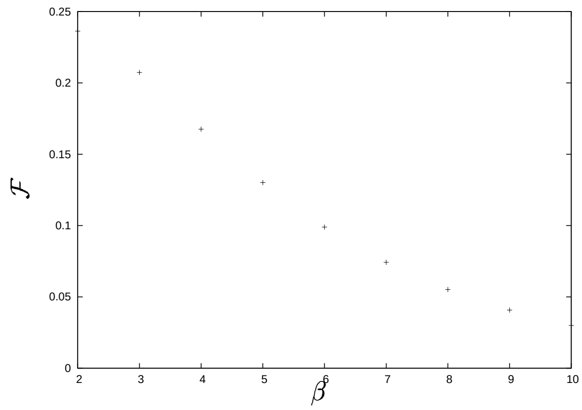


Figure 2. Variation of response function with β, pipe case ($\omega = 0$)

response function $\mathcal{F} = \int_0^\infty |\hat{U}(\xi, y = 1)|d\xi$. Results for the variation of this quantity, as the crossflow wavenumber is varied are shown in figure 1(b), which indicates a maximum at $\beta \approx 1.38$.

There are some interesting similarities and differences in the case of developing flow inside a circular pipe. Firstly if the spanwise wavenumber is replaced by an azimuthal wavenumber β, then this must be restricted to integer values. Second, detailed analysis reveals that $\beta \geq 2$. Again, careful numerical work (including a judicious choice of coordinates) is necessary for accurate stability results. Since (for the permissible values of azimuthal wavenumber) the axial velocity component of the disturbance field is zero, the use of the previous definition of response function is no longer appropriate, and so instead we took $\mathcal{F} = \int_0^\infty |\hat{U}(\xi, r = \sqrt{\frac{1}{2}})|d\xi$, (where r is the non-dimensional radius, such that $0 \leq r \leq 1$). Results for this measure (again, for $\omega = 0$) are shown in figure 2. The largest flow response is seen clearly to occur at the smallest permissible value of azimuthal wavenumber, i.e. $\beta = 2$.

CONCLUSION

Given the significant flow disturbance response that is possible, there is therefore some potential for flow transition to be provoked through the mechanism described in this paper.

References

[1] Luchini, P.: Reynolds-number-independent instability of the boundary layer over a flat surface: optimal perturbations. *J. Fluid Mech* **327** 101, 1996.

[2] Duck, P.W., Stow, S.R. and Dhanak, M.R.: Non-similarity solutions to the corner boundary-layer equations (and the effects of wall transpiration). *J. Fluid Mech* **400** 125, 1999.

[3] Andersson, P., Berggren, M. and Henningson, D.S.: Optimal disturbances and bypass transition in boundary layers. *Phys. Fluids* **11** 134, 1999.

[4] Luchini, P.: Reynolds-number-independent instability of the boundary layer over a flat surface: optimal perturbations. *J. Fluid Mech* **404** 289, 2000.