NAVIER-STOKES TURBULENCE

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Summary Numerical study of Eulerian-Lagrangian analysis of Navier-Stokes turbulence is presented. Near identity transformations associated with diffusive Lagrangian labels capture a time scale of small scale motion. A novel characterization of the singular perturbation nature as $\nu \to 0$ is proposed as connection anomaly.

INTRODUCTION

One of the important properties of the the Navier-Stokes flows is that in the limit of vanishing viscosity their behavior differs from that of corresponding Euler equations. For example, in the inviscid case the total energy is conserved, whereas in slightly viscous Navier-Stokes flows it is observed to be dissipated in a nontrivial fashion. This anomalous behavior, finite energy dissipation in the limit of small viscosity is the fundamental premise of Kolmogorov's theory of turbulence. Another example is the phenomenon of vortex reconnection in slightly viscous flows, as opposed to frozen vortex lines in inviscid fluids.
Recently, a framework of the Navier-Stokes equations suitable for studying the singular perturbation nature and/or topological properties of vortex lines in viscous flow has been developed by one of the authors [1, 2]. It has been applied to numerical simulations of the Navier-Stokes equations and its usefulness for monitoring vortex reconnection has been shown. The purpose of this paper is to apply it to fully-developed turbulence to quantify its anomalous behavior.

THE EULERIAN-LAGRANGIAN FORMULATION

The incompressible Navier-Stokes equations read with standard notations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \nu \triangle \boldsymbol{u}, \quad \mathrm{div}\ \boldsymbol{u} = 0 \tag{1}$$

We recast them using impulse $\boldsymbol{w}$, e.g. [3]

$$\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} = -(\nabla \boldsymbol{u})^T \boldsymbol{w} + \nu \triangle \boldsymbol{w}, \tag{2}$$

where T means matrix transpose. The velocity $\boldsymbol{u}$ is obtained by solenoidal projection $\boldsymbol{u} = \boldsymbol{P}[\boldsymbol{w}]$. The crux of the Eulerian-Lagrangian formalism is the introduction of diffusive Lagrangian label as

$$\frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{A} = \nu \triangle \boldsymbol{A}, \tag{3}$$

and velocity decomposition $\boldsymbol{u} = \boldsymbol{P}[(\nabla \boldsymbol{A})^T \boldsymbol{v}]$, much like Weber formula in the inviscid case.
For consistency with the Navier-Stokes equations, virtual velocity $\boldsymbol{v}$ was shown to obey [1]

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} = 2\nu \boldsymbol{C} : \nabla \boldsymbol{v} + \nu \triangle \boldsymbol{v}, \tag{4}$$

where $i-$th component of $\boldsymbol{C} : \nabla \boldsymbol{v}$ is $C_{m,k;i}\dfrac{\partial v_m}{\partial x_k}$ and $C_{m,k;i} \equiv \dfrac{\partial x_j}{\partial A_i}\dfrac{\partial^2 A_m}{\partial x_j \partial x_k}$. Connection $\boldsymbol{C}$ measures non-commutativity between the Euler and Euler-Lagrange derivatives; $[\nabla_A^i, \nabla_E^k] = C_{m,k;i}\nabla_A^m$. Practically, in stead of $\boldsymbol{v}$, we solve for displacement vector $\boldsymbol{\ell} = \boldsymbol{A} - \boldsymbol{x}$ which satisfies a passive equation of the form

$$\frac{\partial \boldsymbol{\ell}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\ell} = -\boldsymbol{u} + \nu \triangle \boldsymbol{\ell}. \tag{5}$$

Once $\boldsymbol{\ell}$ is obtained, $\boldsymbol{C}$ and all other quantities of interest can be obtained *a posteori* by matrix inversion and by $\nabla \boldsymbol{A} = \nabla \boldsymbol{\ell} + \boldsymbol{I}$.
In inviscid fluids $\boldsymbol{A}$ remains invertible under smooth evolution because $\det(\nabla \boldsymbol{A})$ is constant. In viscous fluids the determinant is not preserved in general. In [4] it was found that the matrix becomes non-invertible in time and this phenomenon is related with vortex reconnection. It is necessary to reset $\boldsymbol{\ell} = 0$ if $|\det(\nabla \boldsymbol{A})| \le \epsilon$, where ϵ is a small parameter.

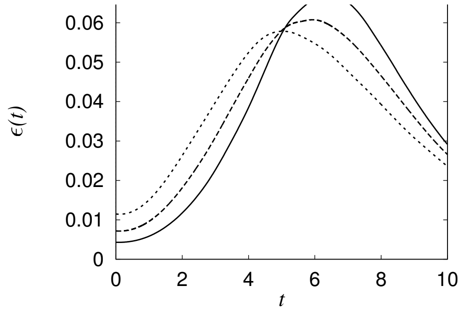

Figure 1: Time evoultion of the energy disspation rate for $\nu = 4 \times 10^{-3}$(dotted), 2.5×10^{-3}(dashed) and 1.5×10^{-3}(solid)

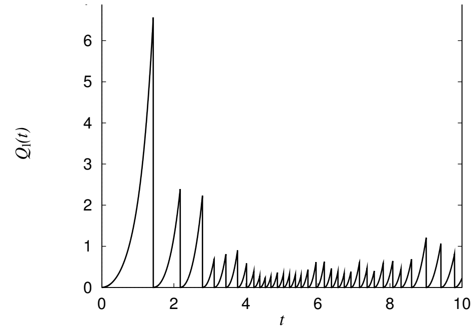

Figure 2: Time evoultion of enstrophy of displacement for the case of $\nu = 2.5 \times 10^{-3}$.

APPLICATION TO DECAYING TURBULENCE

A $2/3-$dealiased pseudo-spectral method was employed under periodic boundary conditions with grid points typically 256^3. Time marching was performed with a fourth-order Runge-Kutta scheme. The initial condition has an energy spectrum of the form

$$E(k) \propto k^2 \exp(-k^2), \text{ with } \left\langle |\boldsymbol{u}|^2 \right\rangle = 1.$$

Numerical parameters used are $\nu = 4 \times 10^{-3}, 2.5 \times 10^{-3}$ and 1.5×10^{-3} with $\Delta t = 2 \times 10^{-3}$ and with $\epsilon = 0.01$. The energy dissipation rate $\epsilon(t)$ attains its maximum around $t = 6$ in Fig.1. We show time development of

$$Q_{\boldsymbol{\ell}}(t) = \frac{1}{2} \left\langle |\nabla \times \boldsymbol{\ell}|^2 \right\rangle.$$

in Fig.2. In the developed stage the resetting time interval is found to be comparable to Kolmogorov's time scale or less. Occurrence of resetting implies that the first term on the RHS of (4) is not negligible in the limit of small viscosity. To quantify the anomaly of connection we are led to the following hypothesis: for consecutive resetting times $t_j,\ t_{j+1}$ there are positive constants A_p independent of ν such that

$$\lim_{\nu \to 0} \nu \int_{t_j}^{t_{j+1}} ||C||_p^2 \mathrm{d}t > A_p > 0, \ \ ||\boldsymbol{C}||_p \equiv \left(\frac{1}{(2\pi)^3} \int |\boldsymbol{C}|^p \boldsymbol{dx} \right)^{1/p}. \tag{6}$$

The nondimensional integral is observed to be bounded from below by positive constants of order 10^{-1}, 10^{-1} and 10^2 for $p = 1, 2$ and ∞, respectively. The connection anomaly is a characterization of singular nature of the limit $\nu \to 0$.

References

[1] Constantin, P. 2001 An Eulerian-Lagrangian Approach to the Navier-Stokes Equations *Commun. Math. Phys.* **216**, 663–686.

[2] Constantin, P. 2003 Near identity transformations for the Navier-Stokes equations in *Handbook of Mathematical Fluid Dynamics* (ed. S. Friedlander, D. Serre). vol. 2, pp. 117–141. Elsevier.

[3] Buttke, T.& Chorin, A. 1993 Turbulence calculations in magnetization variables *Appl. Num. Math.* **12**, 47–54.

[4] Ohkitani, K. & Constantin, P. 2003 Numerical study of the Eulerian-Lagrangian formulation of the Navier-Stokes equations *Phys. Fluids* **15**, 3251–3254.