MATERIAL MODELS FOR HOOKEAN MATERIALS WITH VOIDS OR CRACKS

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Summary: Stress-strain relations for Hookean materials with spherical voids or penny-shaped microcracks are derived. The constitutive relation for voided material is based on the analytical expression by Eshelby whereas the theoretical work by Kachanov provided the foundation for the material model of microcracked material. The postulate of strain equivalence was shown to be incompatible with the analytical expression by Eshelby.

Damage description by the postulate of strain equivalence - the effective stress $\tilde{\boldsymbol{\sigma}}$ concept

By following the postulate of strain equivalence by Chaboche (1978, p. 19) the effective stress tensor $\tilde{\boldsymbol{\sigma}}$ and the constitutive tensor for the damaged material $\tilde{\mathbf{C}}$ can be obtained from

$$\tilde{\boldsymbol{\sigma}} := \mathbf{C}:\boldsymbol{\varepsilon}^{e} \quad \text{and} \quad \boldsymbol{\sigma} := \tilde{\mathbf{C}}:\boldsymbol{\varepsilon}^{e} \quad \text{which give} \quad \tilde{\boldsymbol{\sigma}} := \mathbf{C}:\tilde{\mathbf{S}}:\boldsymbol{\sigma}. \tag{1}$$

Equation $(1)_3$ utilises the definition of the compliance tensor for the damaged material $\tilde{\mathbf{S}}$. The definition is $\tilde{\mathbf{S}}:\tilde{\mathbf{C}} := \mathbf{I}$. The compliance tensor $\tilde{\mathbf{S}}$ is assumed to be separable as follows:

$$\tilde{\mathbf{S}} = \mathbf{S} + \mathbf{S}^{d}, \tag{2}$$

where the superscript "d" refers to damage. Substitution of the Separation (2) into Equation $(1)_3$ yields

$$\tilde{\boldsymbol{\sigma}} = \mathbf{C}:\tilde{\mathbf{S}}:\boldsymbol{\sigma} = \mathbf{C}:[\mathbf{S} + \mathbf{S}^{d}]:\boldsymbol{\sigma} = [\mathbf{I} + \mathbf{C}:\mathbf{S}^{d}]:\boldsymbol{\sigma}. \tag{3}$$

The concept of the effective stress $\tilde{\sigma}$ by Rabotnov (1968, p. 344) can be extended for three-dimensional form as follows:

$$\tilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{1 - D}. \tag{4}$$

In order that Expressions $(1)_3$ and (4) would equal, the following should hold:

$$\mathbf{C}:\tilde{\mathbf{S}} = \mathbf{I}(1 - D)^{-1} \quad \text{which yields} \quad \tilde{\mathbf{S}} = (1 - D)^{-1}\mathbf{S}. \tag{5}$$

Separation (2) allows Equation $(5)_2$ to be written in the form

$$\mathbf{S} + \mathbf{S}^{d} = (1 - D)^{-1}\mathbf{S}, \quad \text{which yields} \quad \mathbf{S}^{d} = \frac{D}{1 - D}\mathbf{S}. \tag{6}$$

Porous material with linear elastic matrix

Eshelby (1957) studied the elastic field in a Hookean material containing an ellipsoidal inclusion. As a special case he determined the value for the complementary strain-energy density w^{c} of a material containing "a volume fraction f" of inhomogeneous spheres. For the purpose of this work the inhomogeneous spheres are "replaced" by spherical cavities. This is done by assuming that the values for the elastic constants for the cavities vanish. The complementary strain-energy density $w^{c}(\boldsymbol{\sigma},f)$ takes the form (Eshelby, 1957, p. 390)

$$w^{c}(\boldsymbol{\sigma},f) = \frac{1}{2}\left[\frac{1}{3(3\lambda + 2\mu)}(1 + Af)[\mathbf{1}:\boldsymbol{\sigma}]^{2} + \frac{1}{2\mu}(1 + Bf)\,\mathbf{s}:\mathbf{s}\right], \tag{7}$$

where λ and μ are the Lamé elastic constants of the matrix material. The values for the parameters A and B for spherical cavities and the deviatoric stress $\mathbf{s}$ and are obtained from

$$A = \frac{6\mu + 3\lambda}{4\mu} \quad \text{and} \quad B = \frac{15(1 - \nu)}{7 - 5\nu} \quad \text{and} \quad \mathbf{s} := \mathbf{K}:\boldsymbol{\sigma} \quad \text{where} \quad \mathbf{K} := \mathbf{I} - \tfrac{1}{3}\mathbf{1}\,\mathbf{1}. \tag{8}$$

The notations $\mathbf{1}$ and $\mathbf{I}$ refer to the second-order and to the fourth-order identity tensor. Based on Expression (7) the elastic strain tensor $\boldsymbol{\varepsilon}^{e}$ takes the following form:

$$\boldsymbol{\varepsilon}^{e} = \frac{\partial w^{c}(\boldsymbol{\sigma},f)}{\partial \boldsymbol{\varepsilon}^{e}} = \frac{1}{3(3\lambda + 2\mu)}(1 + Af)\,\mathbf{1}\,\mathbf{1}:\boldsymbol{\sigma} + \frac{1}{2\mu}(1 + Bf)\,\mathbf{K}:\boldsymbol{\sigma}. \tag{9}$$

Expression (9) can be written in the form

$$\boldsymbol{\varepsilon}^{e} = \left[\mathbf{S} + \mathbf{S}^{v}(f)\right]:\boldsymbol{\sigma}, \tag{10}$$

where the compliance tensor for Hookean material $\mathbf{S}$ and the compliance tensor voids $\mathbf{S}^{v}(f)$ are defined by

$$\mathbf{S} := \frac{1}{3(3\lambda + 2\mu)}\mathbf{1}\,\mathbf{1} + \frac{1}{2\mu}\mathbf{K} \quad \text{and} \quad \mathbf{S}^{v}(f) := \frac{Af}{3(3\lambda + 2\mu)}\mathbf{1}\,\mathbf{1} + \frac{Bf}{2\mu}\mathbf{K}. \tag{11}$$

Equation $(6)_2$ shows that according to the postulate of strain equivalence the compliance tensors $\mathbf{S}$ and $\mathbf{S}^{d}$ have a linear relationship. Comparison of Definitions (11) hows that material parameters A and B have to equal in order the tensor $\mathbf{S}^{\mathrm{v}}(f)$ would be possible to express as a function of the tensor $\mathbf{S}$. If the Poisson's ratio took the value of $\nu = 0.2$. Therefore, for steels, for example, the postulate of strain equivalence does not give compatible results with the analytical expression.

Hookean material with micricracks

According to Kachanov (1980, p. 1045) the complementary strain-energy density w^{c} of a Hookean solid with non-interactive cracks is given by

$$w^{\mathrm{c}}(\boldsymbol{\sigma},\ldots) = \frac{1}{2}\left[\frac{1}{3(3\lambda+2\mu)}[\mathbf{1}:\boldsymbol{\sigma}]^2 + \frac{1}{2\mu}\mathbf{s}:\mathbf{s}\right] + \frac{1}{\mathrm{V}^{\mathrm{b}}}\sum_{p=1}^{k}\frac{1}{2}\int_{A^p}(\vec{n}^p\cdot\boldsymbol{\sigma})\cdot\vec{b}^p\,\mathrm{d}A^p, \tag{12}$$

where V^{b} is the volume of the body in the initial crack free configuration and k is the number of cracks within the volume V^{b} (Vakulenko and Kachanov, 1971, p. 160 and 161). In Equation (12) the area of the p'th crack is denoted by A^p and the unit normal vector to the surface A^p is denoted by $\vec{n}^p$. The vector $\vec{b}^p$ describing the jump between the material points across the crack is, according to Kachanov [1980, Eq. (20a)], for penny-shaped cracks

$$\vec{b}^{p\mathrm{K}} = B_1\sqrt{(a^p)^2 - r^2}\left\{\boldsymbol{\sigma}\cdot\vec{n}^p - \frac{\nu}{2}\vec{n}^p(\vec{n}^p\cdot\boldsymbol{\sigma}\cdot\vec{n}^p)\right\}, \qquad \text{where} \qquad B_1 = \frac{16(1-\nu^2)}{\pi E(2-\nu)}. \tag{13}$$

In Expression (13) a^p is the radius of the p'th microcrack, r is the radial coordinate of the microcrack. The superscript "K" in $\vec{b}^{p\mathrm{K}}$ refers to Kachanov. In Equation (13) a misprint is corrected i.e. the first term between the braces $\vec{n}^p\cdot\boldsymbol{\sigma}$ by Kachanov is replaced here by the term $\boldsymbol{\sigma}\cdot\vec{n}^p$. In this work Kachanov's Equation (13) [already corrected] is replaced by

$$\vec{b}^p = B_1\sqrt{(a^p)^2 - r^2}\left\{\boldsymbol{\sigma}\cdot\vec{n}^p - [1-H]\,\vec{n}^p(\vec{n}^p\cdot\boldsymbol{\sigma}\cdot\vec{n}^p) - \frac{\nu}{2}H\vec{n}^p(\vec{n}^p\cdot\boldsymbol{\sigma}\cdot\vec{n}^p)\right\}, \quad \text{where} \quad H = H(\vec{n}^p\cdot\boldsymbol{\sigma}\cdot\vec{n}^p). \tag{14}$$

The Heaviside function $H(\boldsymbol{\sigma},\vec{n}^p)$ guarantees that under compression the microcrack surfaces do not penetrate each other. Due to simplicity a uniaxial microcrack field is studied. The normals of the microcracks are assumed to be parallel to the x_1-axis. The stress-strain relation has the same form as for porous material, viz.

$$\boldsymbol{\varepsilon}^{\mathrm{e}} = \left[\mathbf{S} + \mathbf{S}^{\mathrm{c}}\right]:\boldsymbol{\sigma}. \tag{15}$$

If the number of parallel microcracks is denoted by ρ_{ij}, the non-zero components of the compliance tensor $\mathbf{S}^{\mathrm{c}}$ are

$$S^{\mathrm{c}}_{1111} = 4B\left(1 - \frac{\nu}{2}\right)\sum_{q=1}^{\rho_{ij}} A_q^{3/2}H(\sigma_{11}), \qquad \text{where} \qquad B = \frac{8(1-\nu^2)}{3\pi\sqrt{\pi}\,(2-\nu)E} \tag{16}$$

and

$$S^{\mathrm{c}}_{1212} = S^{\mathrm{c}}_{1221} = S^{\mathrm{c}}_{2112} = S^{\mathrm{c}}_{2121} = S^{\mathrm{c}}_{1313} = S^{\mathrm{c}}_{1331} = S^{\mathrm{c}}_{3113} = S^{\mathrm{c}}_{3131} = B\sum_{q=1}^{\rho_{ij}} A_q^{3/2}. \tag{17}$$

Discussion and conclusions

The above given stress-strain relations for Hookean material with spherical voids or penny-shaped microcracks play important role in damage mechanics, since they are based on analytical expressions. It is noteworthy that the results of the postulate of strain energy are in contradiction with the analytical stress-strain relation by Eshelby (1957). A keen reader on this topic may study the papers Santaoja (1989) and (2002).

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