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INCREMENTAL ENERGY MINIMIZATION IN MATERIAL INSTABILITY PROBLEMS

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Summary Recent theoretical developments of the incremental energy minimization and its novel applications to material instability problems in time-independent inelastic solids are presented. Necessity of imposing a symmetry restriction on the constitutive law is discussed. The approach yields a natural criterion of selection of the post-critical deformation pattern and provides a computational method for determining deformation paths with automatic branch switching.

INTRODUCTION

The idea of determining the response of an inelastic body to slowly varying loading by minimizing *incrementally* the energy supply to the examined system received recently a wider attention. The concept was originally developed in relation to stability of solution paths in time-independent plasticity at finite deformation, cf. the first author's sectional lecture at IUTAM Congress in Kyoto, 1996 [1]. The theory provided a basis for the computational method that employs non-convex minimization of the incremental energy functional defined on a set of kinematically admissible fields. This method has found applications in calculations of microstructure formation in incrementally nonlinear materials and in crossing multiple bifurcation points with automatic selection of the post-critical deformation path [2].

The aim of this lecture is to provide recent theoretical developments of the incremental energy minimization and its novel applications to material instability problems in inelastic solids.

THEORETICAL DEVELOPMENTS

In outline, the incremental energy minimization in isothermal quasi-static deformation is characterized as follows

$$\Delta E \to \min \qquad \text{subject to kinematical constraints,} \qquad \Delta E = \Delta W + \Delta\Omega\,, \tag{1}$$

where a prefix Δ denotes a virtual increment corresponding to a non-zero increment of a loading parameter λ, ΔW is the total work increment supplied to the deforming body, and $\Delta\Omega$ is an increment in potential energy of the loading device (assumed conservative). It has been shown that an exact incremental solution represented by a velocity field $\tilde{\mathbf{v}}$ is found by the minimization (1) applied to the part $\Delta_2 E$ that contains only the *second-order* terms with respect to a time increment Δt, reduced to

$$\int_B U(\dot{\mathbf{F}})\,dV\; + \ldots \to \min \qquad U = \tfrac{1}{2}\dot{\mathbf{S}}\cdot\dot{\mathbf{F}}\,, \qquad \dot{\mathbf{S}} = \partial U/\partial\dot{\mathbf{F}}\,, \qquad \dot{\mathbf{F}} = \mathrm{Grad}\,\mathbf{v}\,, \qquad \tilde{\mathbf{v}} \in \mathcal{V}\,, \tag{2}$$

where U is the constitutive potential for determining the nominal stress rate $\dot{\mathbf{S}}^T$ from the forward rate $\dot{\mathbf{F}}$ of deformation-gradient $\mathbf{F}$, and $\mathcal{V}$ is a set of kinematically admissible velocity fields on a reference domain B. In (2), only the leading term (corresponding to $\Delta_2 W$) is displayed, while the second-order terms corresponding to $\Delta_2\Omega$ are indicated by dots. Existence of a potential U is required for a rigorous justification of the second-order minimization approach. The essence of (2) is that it represents *non-convex* minimization. In result, the modern *relaxation methods* (quasi-convexification and rank-one-convexification) developed in the context of elastic energy minimization are extended in a natural way to material instabilities in a class of inelastic solids. On using the approximation $\Delta\mathbf{F} = \mathrm{Grad}\,\mathbf{v}\,\Delta t + \ldots$ and homogeneity of degree two of U, (2) can be used to generate finite increments as in some recent papers by other authors, e.g. [3]. However, finite time-step solutions are merely approximate unless further constitutive assumptions are introduced.

In a thermodynamic framework with internal variables $\boldsymbol{\alpha}$, the incremental response of the material can be determined from the following minimization problem derived from (1)

$$J(\dot{\boldsymbol{\alpha}}) = \tfrac{1}{2}\dot{\boldsymbol{\alpha}}\cdot\phi_{,\boldsymbol{\alpha}\boldsymbol{\alpha}}\cdot\dot{\boldsymbol{\alpha}} + \tfrac{1}{2}D_1(\dot{\boldsymbol{\alpha}},\dot{\boldsymbol{\alpha}}) + \dot{\mathbf{F}}\cdot\phi_{,\mathbf{F}\boldsymbol{\alpha}}\cdot\dot{\boldsymbol{\alpha}} \to \min\,, \qquad D_1(\dot{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}}) = \hat{\boldsymbol{\alpha}}\cdot D_{,\boldsymbol{\alpha}}(\dot{\boldsymbol{\alpha}},\boldsymbol{\alpha}) \tag{3}$$

where ϕ is the Helmholtz free energy density, $D = D(\dot{\boldsymbol{\alpha}},\boldsymbol{\alpha})$ is the dissipation function, and $\dot{\boldsymbol{\alpha}}$ lies within the cone $\mathcal{L} = \{\dot{\boldsymbol{\alpha}}:\; D(\dot{\boldsymbol{\alpha}},\boldsymbol{\alpha}) = -\phi_{,\boldsymbol{\alpha}}\cdot\dot{\boldsymbol{\alpha}}\}$. As in (2), the essence of (3) lies in non-convex minimization. It has recently been shown [4] that the symmetry restriction $D_1(\dot{\boldsymbol{\alpha}},\dot{\boldsymbol{\alpha}}+\delta\dot{\boldsymbol{\alpha}}) = D_1(\dot{\boldsymbol{\alpha}}+\delta\dot{\boldsymbol{\alpha}},\dot{\boldsymbol{\alpha}})$ for $\dot{\boldsymbol{\alpha}} \pm \delta\dot{\boldsymbol{\alpha}} \in \mathcal{L}$ must be imposed on the dissipation function as a condition necessary for the intrinsic consistency between the first- and second-order minimization in (1). Under this symmetry restriction and in the internal-variable formulation of multi-mode inelasticity, in the present paper theorems are formulated and proven that provide a novel justification of the second-order version of (1) as the energy condition necessary for stability of a solution path.

NOVEL APPLICATIONS

Three examples of application of (1), not published so far, are presented and discussed.

Of special interest is the case when instability of a uniform deformation path leads to the formation of a microstructure in an initially homogeneous material. This is illustrated by simulation of the formation of initially one and later two families of shear bands, cf. Fig 1, in an incrementally nonlinear elastoplastic material subjected to non-proportional straining. The results to be presented in the lecture provide an extension of those in [2] by incorporating the effect of an angular path of deformation. A particular feature of the obtained deformation pattern is that its formation is accompanied by multiple bifurcations. The step-by-step use of non-convex minimization makes it possible to follow the complex deformation path with automatic selection of the post-bifurcation solutions.

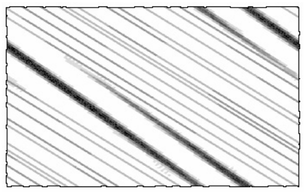

Fig. 1. Two intersecting families of shear bands in an incrementally nonlinear material, initially homogeneous.

The second example is concerned with rapid formation of shear bands that carry large plastic strains. While in the example mentioned above the shear bands develop gradually, in this example a single band forms instantaneously in the time scale of a loading program ($\lambda(t)$). It is shown how the incremental energy minimization (1) can be used to predict the onset of formation of such bands as well as their orientation.

In another example, bifurcation of the layered pattern of martensitic phase transformation in a shape memory alloy is analyzed. The upper pattern (I) sketched (not to scale) in Fig. 2 corresponds to macroscopically uniform distribution of internally twinned martensite plates, while in an alternative solution (II) the martensite plates first appear within a certain volume fraction only, forming an evolving rank-three laminated microstructure. Performed calculations have shown that the pattern (II) is energetically preferable initially, in the sense of the minimization rule (1).

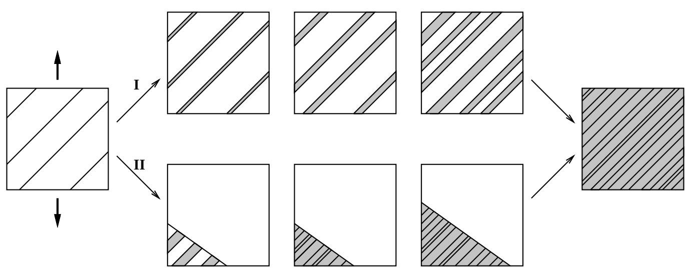

Fig. 2. Bifurcation of the layered transformation pattern: the transformation can proceed quasi-uniformly (upper pattern I) or be completed first within some volume fraction of the material (lower pattern II). [5]

CONCLUSIONS

The non-convex minimization of incremental energy

– is applicable to a broad class of time-independent inelastic solids, but requires a *symmetry assumption*,

– yields a natural *criterion of selection* of the material response or deformation pattern,

– provides a *computational method* for determining deformation paths with automatic branch switching.

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