NEAR CRITICAL UNSTEADY TREE-DIMENSIONAL TRIPLE DECK FLOWS

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Summary Near critical, i.e. transitional laminar separation bubbles are studied using triple deck theory. The investigation is based on a local bifurcation analysis associated with the occurrence of non-unique solutions in the corresponding planar steady flow calculations. This leads to a substantial simplification of the primary problem formulation allowing a largely analytic description of the evolution of unsteady three-dimensional perturbations.

MOTIVATION

Earlier investigations of steady two-dimensional marginally separated laminar boundary layers [1], [2] in the limit of high Reynolds-numbers *Re* have shown that the non-dimensional wall shear (or equivalently the negative non-dimensional perturbation displacement thickness) is governed by a nonlinear integro-differential equation. This equation contains a single *controlling parameter* Γ characterizing, for example, the angle of attack of a slender airfoil and has the important property that (real) solutions exist up to a *critical value* Γ_c of Γ only. Recent investigations [3] of three-dimensional unsteady perturbations of such a steady two-dimensional marginally separated laminar boundary layer with special emphasis on the flow behaviour near Γ_c have shown that the integro-differential equation which governs these disturbances if $\Gamma_c - \Gamma = O(1)$ reduces to a nonlinear partial differential equation - known as (forced) *Fisher equation* - as Γ approaches the critical value Γ_c. The bifurcation analysis of this problem associated with the *non-uniqueness* of the steady planar flow thus leads to a significant simplification of the problem allowing, among others, a systematic study of the application of devices used in boundary layer control and an analytical analysis of the conditions leading to the formation of *finite time singularities* which have been observed in earlier numerical studies of unsteady two-dimensional and three-dimensional flows. Also it was found possible to construct exact solutions which describe waves of constant form travelling in the spanwise direction. These waves may contain singularities which can be interpreted as vortex sheets. The existence of these solutions strongly suggests that solutions of the Fisher equation which lead to finite time blow-up can be extended *beyond* the blow-up time thereby generating *moving singularities* which can be interpreted as vortical structures qualitatively similar to those emerging in direct numerical simulations of transitional laminar separation bubbles. This was further supported by asymptotic analysis. Additionally, it could be shown that finite time singularities describing the phenomenon of bubble bursting may occur in below-critical as well as above-critical situations. However, while this phenomenon requires a certain finite perturbation level if $\Gamma < \Gamma_c$ it is triggered by even infinitesimally small disturbances if $\Gamma > \Gamma_c$ where also *self-sustained oscillations* with periodically repeated bubble bursts ('vortex shedding') are possible. Multiplicity of solutions and critical values of the controlling parameter beyond which steady state solutions do not exist are a characteristic feature of marginally separated flows. However, similar phenomena do occur also in situations where triple-deck theory applies, i.e. in situations where a fully attached boundary layer is forced to separate due to the presence of a large adverse pressure gradient. Examples displaying such a branching behaviour include supersonic flows past flared cylinders, [4], subsonic flows past compression ramps and subsonic trailing edge flows, [5]. Furthermore, in a recent publication [6] it was argued that bursting processes in transitional boundary layer flows share common properties which do not depend on the specific problem under consideration. This suggests that an analysis similar to the one carried out for marginally separated boundary layers should be possible also in the context of classical triple deck theory which is the aim of the present paper.

SHORT OUTLINE

Specifically, we consider the example of the unsteady extension of subsonic planar ramp flow. Boundary layer separation and non-uniqueness of the steady problem was found for negative ramp angles (convex corners), see e.g. [5]. Let t, x, y and u, v denote suitably scaled quantities: time, coordinates in streamwise, wall normal direction and the corresponding velocity components. The governing equations for the flow field in the lower deck of the interaction region located at the corner supplemented with the interaction law then take the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}\,, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \quad p = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\partial(A-f)/\partial\xi}{\xi - x}\,\mathrm{d}x \tag{1}$$

with the boundary conditions $y = 0$: $u = v = 0$, $y \to \infty$: $u \sim y + A$, $x \to -\infty$: $u \sim y$. Here p is the pressure, A the displacement function and f the local shape of the (smooth) corner. Near the critical value $\alpha_c < 0$ of the ramp angle α the observed branching behaviour suggests a parabolic approximation of the corresponding field quantities:

$$\begin{aligned}(u,v) &\sim (u_c,v_c)(x,y) + \varepsilon^2(u_1,v_1)(x,y,\bar{t}) + \varepsilon^4(u_2,v_2)(x,y,\bar{t}) + \cdots\,,\\ (p,A) &\sim (p_c,A_c)(x) + \varepsilon^2(p_1,A_1)(x,\bar{t}) + \varepsilon^4(p_2,A_2)(x,\bar{t}) + \cdots\,, \quad f = f_c(x) + \varepsilon^4 f_2(x,\bar{t})\end{aligned} \tag{2}$$

where we have introduced the perturbation parameter $\varepsilon^4 := \alpha - \alpha_c \to 0$ and the appropriate scaling $\bar{t} = \varepsilon^2 t$.
The leading order problem $O(\varepsilon^0)$ is simply given by the steady version of (1) at the branching point $\alpha = \alpha_c$. Higher order contributions $O(\varepsilon^{2i})$, $i = 1, 2$ can be written in the form $M\vec{r}_i = \vec{b}_i$, where M denotes a singular linear operator matrix, $\vec{r}_i = (u_i, v_i, p_i, A_i)$ the flow field perturbations and $\vec{b}_1 = \vec{0}$, $\vec{b}_2(\vec{r}_1, f_2)$ the right hand sides. Since M affects the x and y dependency of the field quantities only, one deduces $\vec{r}_1(x, y, \bar{t}) = c(\bar{t})\,\vec{r}(x, y)$. Furthermore, the right eigenfunction $\vec{r}$ of M can be expressed in terms of a single function, the 'perturbation' stream function ψ:

$$\vec{r} = \left(\psi_y, -\psi_x, -\lim_{y\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\psi_{y\xi}}{\xi - x}\, \mathrm{d}\xi,\ \lim_{y\to\infty} \psi_y\right) . \tag{3}$$

The evolution equation for the shape function $c(\bar{t})$ is obtained from the solvability condition (Fredholm alternative) at $O(\varepsilon^4)$, which finally yields

$$\frac{\mathrm{d}c}{\mathrm{d}\bar{t}} + \mu c^2 = \pm \bar{g}(\bar{t})\ , \quad \alpha \gtrless \alpha_c\ . \tag{4}$$

Here $\mu > 0$ and $\bar{g}$ denote a constant and a forcing term associated with the evaluation of scalar products of left eigenfunctions of M and the right hand side $\vec{b}_2$. The special case of unforced flow characterized by $f_2(x, \bar{t}) = f_2(x)$, $\bar{g}(\bar{t}) = \delta = const > 0$ and subjected to the initial condition $c(\bar{t}_0) = c_0$ allows for an analytical solution of (4)

$$\frac{c}{c_s} = \frac{c_0 + c_s \tanh[\sqrt{\mu\delta}\,(\bar{t} - \bar{t}_0)]}{c_s + c_0 \tanh[\sqrt{\mu\delta}\,(\bar{t} - \bar{t}_0)]}\ , \quad \alpha > \alpha_c\ , \qquad \frac{c}{c_s} = \frac{c_0 - c_s \tan[\sqrt{\mu\delta}\,(\bar{t} - \bar{t}_0)]}{c_s + c_0 \tan[\sqrt{\mu\delta}\,(\bar{t} - \bar{t}_0)]}\ , \quad \alpha < \alpha_c\ . \tag{5}$$

The stationary points $\pm c_s$, $c_s = \sqrt{\delta/\mu}$ of (4) correspond to steady lower and upper branch solutions in the below-critical case $\alpha > \alpha_c$, Fig. 1(a), whereas *no* steady state exists for the above-critical regime $\alpha < \alpha_c$, Fig. 1(b). Finite time singularities are seen to develop if $c_0/c_s < -1$ under below-critical, but are found to be inevitable under above-critical flow conditions. Most interestingly, in both cases the corresponding solution can be continued *beyond* the blow-up time. In this context the important question arises, how these singularities can be *resolved* where the present theory breaks down locally by taking into account effects which have been neglected so far. The time period of self-sustained vortex shedding associated with above-critical flow conditions conveniently can be written in terms of the Strouhal-number

$$\mathit{Str} \propto \mathit{Re}^{1/4}(\alpha_c - \alpha)^{1/2}\ , \quad \mathit{Re} \to \infty\ ,\ (\alpha_c - \alpha) \to 0^+\ . \tag{6}$$

As can be shown, incorporation of three-dimensional effects indeed lead to nonlinear evolution equations of Fisher's type as in the case of marginally separated boundary layer flows. The properties of their solutions will be discussed in detail.

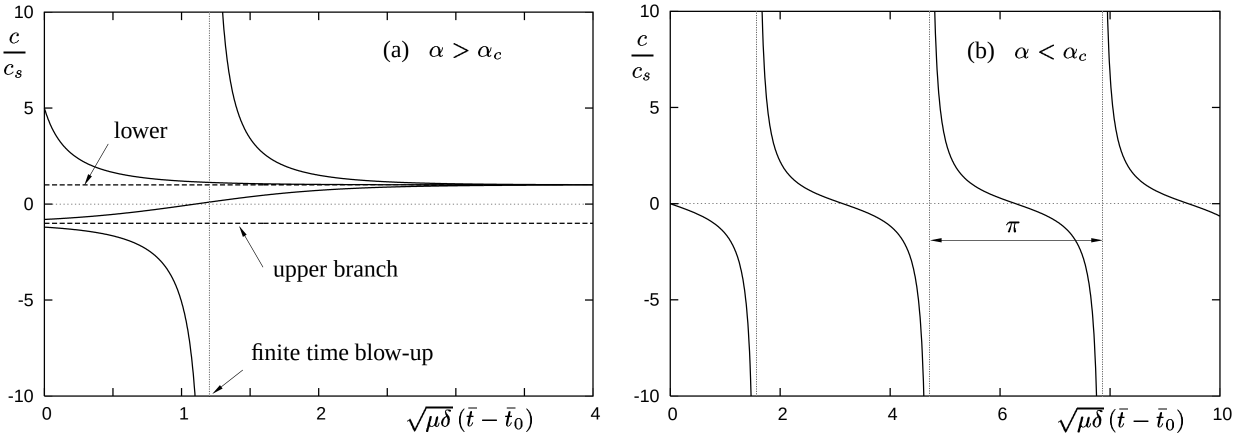

Figure 1: (a) Perturbations of *stable* lower and *unstable* upper branch solutions. (b) Self-sustained bubble bursting.

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