PSEUDO-RIGID BODIES VIEWED AS GLOBALLY CONSTRAINED CONTINUA

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Summary Pseudo-rigid continua can maintain homogeneous deformations in the presence of arbitrary applied loads. Their ability to do so can be explained by treating them as globally constrained continua. A system of indeterminate reactive stresses, analogous to those in a rigid body, are called into play to maintain the homogeneity of the deformation. The governing equations are deduced and a set of Lagrange's equations are also established.

Bodies assumed to be capable of keeping their deformation fields homogeneous have been studied extensively. Initiated by Slawianowski [1, 2], this field was elaborated by Cohen [3] and by Cohen & Muncaster [4, 5]. The theory of elastic pseudo-rigid bodies has enjoyed considerable success (see e.g., [6, 7]). Rubin [8] introduced the concept of a Cosserat point, which is a particle to which are attached deformable directors satisfying appropriate balance laws. The Cosserat point has been investigated in several studies, e.g., [9, 10]. A Cosserat point with three directors is isomorphic to a pseudo-rigid body.

In the present treatment, pseudo-rigid bodies are treated as 3-dimensional continua that can undergo only homogeneous deformations, no matter what loads are applied to them. Adopting the viewpoint of Antman & Marlow [11] and Marlow [12], we say that a pseudo-rigid body is *globally constrained.* As in the case of ordinary internal constraints, some system of indeterminate reaction stresses must exist to maintain the global constraint. Global material constraints arise in the theories of plates, shells, and rods whenever the 3-dimensional deformation field is assumed to have a restricted form. Podio-Guidugli [13] recognized the essential role that reaction stresses play in the theory of linearly elastic plates. The pseudo-rigid continuum may be viewed as an idealized reinforced body in which the Cauchy stress tensor $\mathbf{T}$ at each material point is the sum of an "active stress" $\mathbf{T}^{(A)}$, which is specified by a constitutive equation, and a "reactive stress" $\mathbf{T}^{(R)}$, which takes on whatever values are necessary to maintain homogeneity of the deformation field [14]. Remarkably, the active stresses in a pseudo-rigid body form an equilibrated system, while the reactive stresses play exactly the same role as in rigid body dynamics. It is shown that the governing equations of the pseudo-rigid continuum $\mathcal{B}$ can be written in the form

$$\mathbf{f} = m\ddot{\bar{\mathbf{x}}}\ , \quad \mathbf{M}^{(A)} = V\bar{\mathbf{T}}\ , \quad \widetilde{\mathbf{M}} = \mathbf{A}\mathbf{E}\ ,$$

where

$m =$ mass of $\mathcal{B}$, $\rho =$ density of $\mathcal{B}$ at time t , $V =$ volume of $\mathcal{B}$, $S =$ boundary of $\mathcal{B}$,

$\bar{\mathbf{x}} =$ position vector of mass center of $\mathcal{B}$, $\mathbf{f} =$ resultant external force acting on $\mathcal{B}$,

$\bar{\mathbf{T}} =$ mean Cauchy stress in $\mathcal{B}$, $\mathbf{E} =$ Euler tensor of $\mathcal{B}$ with respect to the mass center ,

$\mathbf{F} =$ deformation gradient , $\boldsymbol{\pi} =$ position vector relative to the mass center ,

$\mathbf{L} =$ velocity gradient $= \dot{\mathbf{F}}\mathbf{F}^{-1}$, $\mathbf{A} =$ acceleration gradient tensor $= \ddot{\mathbf{F}}\mathbf{F}^{-1}$,

$\mathbf{M}^{(A)} = \int_S \mathbf{t}^{(A)} \otimes \boldsymbol{\pi}\, da =$ Möbius (or "astatic") tensor of active traction field $\mathbf{t}^{(A)}$,

$\widetilde{\mathbf{M}} =$ Möbius tensor of reactive traction field and body force field.

The kinetic energy of $\mathcal{B}$ is the sum of two terms, namely

$$\bar{T} = \frac{1}{2} m \dot{\bar{\mathbf{x}}} \cdot \dot{\bar{\mathbf{x}}} \ , \quad T^\star = \frac{1}{2} \operatorname{tr}(\mathbf{L}\mathbf{E}\mathbf{L}^T) \ .$$

Then,

$$\widetilde{\mathbf{M}} \cdot \mathbf{L} = \dot{T}^\star \ .$$

Following a geometrical procedure utilized by Casey [15] for rigid bodies, it is demonstrated how a pseudo-rigid body can be represented by an abstract particle moving in a 12-dimensional Euclidean space, with points $(\bar{\mathbf{x}}(\eta^1, \eta^2, \eta^3), \mathbf{F}(\xi^1, \dots, \xi^9))$, and whose metric is determined by the radius of gyration of the body. A set of Lagrange's equations for the pseudo-rigid body are derived (without any appeal to virtual work or variational principles):

$$\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \dot{\eta}^\gamma}\right) - \frac{\partial \bar{T}}{\partial \eta^\gamma} = \mathbf{f} \cdot \frac{\partial \dot{\bar{\mathbf{x}}}}{\partial \dot{\eta}^\gamma} \ , \quad \frac{d}{dt}\left(\frac{\partial T^\star}{\partial \dot{\xi}^\gamma}\right) - \frac{\partial T^\star}{\partial \xi^\gamma} = \widetilde{\mathbf{M}} \cdot \frac{\partial \mathbf{L}}{\partial \dot{\xi}^\gamma} \ .$$

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