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CONVOLUTION QUADRATURE BASED BOUNDARY ELEMENT METHOD FOR QUASI-STATIC POROELASTICITY

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Summary Applying the usual spatial discretization on the poroelastic quasi-static integral equations and using the Convolution Quadrature Method for the temporal discretization yields a boundary element time-stepping procedure. The proposed methodology is tested with an example for consolidation processes in a poroelastic half space. The algorithm shows no stability problems and behaves well over a broad range of time step sizes.

INTRODUCTION

Convolution Quadrature Method (CQM)-based Boundary Element formulations are up to now used only in dynamic formulations [3]. The main difference to usual time-stepping BE formulations is the way to solve the convolution integral appearing in most time-dependent integral equations. In the CQM formulation, this convolution integral is approximated by a quadrature rule whose weights are determined by the Laplace transformed fundamental solutions and a multi-step method.
For quasi-static problems in poroelasticity there is no need to apply the CQM because time-dependent fundamental solutions are available. However, these fundamental solutions are highly complicated yielding to very sensitive algorithms. Therefore, it is promising to apply the CQM also on the quasi-static integral equations in poroelasticity (details see [4]).

GOVERNING EQUATIONS

Following Biot's approach [1] to model the behavior of porous media, the governing set of differential equations for the unknowns solid displacement u_i and pore pressure p reads

$$Gu_{i,jj} + \left(K + \frac{1}{3}G\right) u_{j,ij} - \alpha p_{,i} = -F_i \qquad \kappa p_{,ii} - \frac{1}{M}\frac{\partial}{\partial t} p - \alpha \frac{\partial}{\partial t} u_{i,i} = -a \,, \tag{1}$$

with the assumption of a linear elastic skeleton and the fluid flow modeled by Darcy's law. The permeability is denoted by κ and the porosity by ϕ. The bulk material is defined by the shear modulus G and the compression modulus K known from elasticity. Biot's effective stress coefficient α and M complete the set of material parameters. The load vector consists of the bulk body forces F_i and a source in the interstitial fluid a.
The respective integral equation necessary to establish a BE formulation is achieved by formulating the weighted residual statement of the governing differential equations (1) with the matrix of fundamental solutions as weights. The usual procedure, i.e., two integrations by part with respect to the spatial variable $\mathbf{x}$ and one with respect to the time variable t, results in a boundary integral equation. Fortunately, the strong singular behavior of the fundamental solutions T^s_{ij} and Q^f is equal to the elastostatic or acoustic fundamental solutions, respectively. All other fundamental solutions are weakly singular or regular. Hence, the limit to the boundary Γ yields

$$\begin{bmatrix} c_{ij}(\mathbf{y}) & 0 \\ 0 & c(\mathbf{y}) \end{bmatrix} \begin{bmatrix} u_i(t,\mathbf{y}) \\ p(t,\mathbf{y}) \end{bmatrix} = \int_\Gamma \begin{bmatrix} U^s_{ij}(t,\mathbf{y},\mathbf{x}) & -P^s_j(t,\mathbf{y},\mathbf{x}) \\ U^f_i(t,\mathbf{y},\mathbf{x}) & -P^f(t,\mathbf{y},\mathbf{x}) \end{bmatrix} * \begin{bmatrix} t_i(t,\mathbf{x}) \\ q(t,\mathbf{x}) \end{bmatrix} \mathrm{d}\Gamma - \oint_\Gamma \begin{bmatrix} T^s_{ij}(t,\mathbf{y},\mathbf{x}) & Q^s_j(t,\mathbf{y},\mathbf{x}) \\ T^f_i(t,\mathbf{y},\mathbf{x}) & Q^f(t,\mathbf{y},\mathbf{x}) \end{bmatrix} * \begin{bmatrix} u_i(t,\mathbf{x}) \\ p(t,\mathbf{x}) \end{bmatrix} \mathrm{d}\Gamma \,, \tag{2}$$

where the integral free terms c_{ij} and c are those known from elastostatics and acoustics, respectively. The time convolution defined in (3) is denoted by $*$ and all capital letters denote fundamental solutions.
Next, the boundary surface Γ is discretized by E iso-parametric elements Γ_e where F polynomial shape functions $N^f_e(\mathbf{x})$ are defined, i.e., ansatz functions are used to approximate the spatial behavior of the boundary states. Further, a time discretization has to be introduced. Instead of using the time-dependent fundamental solutions, here, the convolution quadrature method proposed by Lubich [2]

$$f * g = \int_0^t f(t-\tau)\, g(\tau)\, \mathrm{d}\tau \approx \sum_{k=0}^{n} \omega_{n-k}\left(\hat{f},\Delta t\right) g(k\Delta t) \quad \text{with} \;\; \omega_n\left(\hat{f},\Delta t\right) = \frac{\mathscr{R}^{-n}}{L} \sum_{\ell=0}^{L-1} \hat{f}\left(\frac{\gamma\left(\mathscr{R} e^{i\ell\frac{2\pi}{L}}\right)}{\Delta t}\right) e^{-in\ell\frac{2\pi}{L}} \tag{3}$$

is used as a promising alternative. In eq. (3), $\hat{f}$ denotes the Laplace transformed function of f and the underlying multi-step method is characterized by the quotient of the generating polynomials $\gamma(z)$.
Hence, after dividing time period t in N time steps of equal duration Δt, so that $t = N\Delta t$, the convolution integrals between the fundamental solutions and the boundary states in (2) are approximated by the convolution quadrature method, i.e.,

the quadrature formula (3) is applied to the integral equation (2). This results in the following boundary element time stepping procedure ($n = 0, 1, \ldots, N$)

$$
\begin{aligned}
\begin{bmatrix} c_{ij}(\mathbf{y}) & 0 \\ 0 & c(\mathbf{y}) \end{bmatrix} \begin{bmatrix} u_i(\mathbf{y}, n\Delta t) \\ p(\mathbf{y}, n\Delta t) \end{bmatrix} = \sum_{e=1}^{E} \sum_{f=1}^{F} \sum_{k=0}^{n} \Bigg\{ & \begin{bmatrix} \omega_{n-k}^{ef}\left(\hat{U}_{ij}^{s}, \mathbf{y}, \Delta t\right) & -\omega_{n-k}^{ef}\left(\hat{P}_{j}^{s}, \mathbf{y}, \Delta t\right) \\ \omega_{n-k}^{ef}\left(\hat{U}_{i}^{f}, \mathbf{y}, \Delta t\right) & -\omega_{n-k}^{ef}\left(\hat{P}^{f}, \mathbf{y}, \Delta t\right) \end{bmatrix} \begin{bmatrix} t_i^{ef}(k\Delta t) \\ q^{ef}(k\Delta t) \end{bmatrix} \\
& - \begin{bmatrix} \omega_{n-k}^{ef}\left(\hat{T}_{ij}^{s}, \mathbf{y}, \Delta t\right) & \omega_{n-k}^{ef}\left(\hat{Q}_{j}^{s}, \mathbf{y}, \Delta t\right) \\ \omega_{n-k}^{ef}\left(\hat{T}_{i}^{f}, \mathbf{y}, \Delta t\right) & \omega_{n-k}^{ef}\left(\hat{Q}^{f}, \mathbf{y}, \Delta t\right) \end{bmatrix} \begin{bmatrix} u_i^{ef}(k\Delta t) \\ p^{ef}(k\Delta t) \end{bmatrix} \Bigg\}
\end{aligned}
\tag{4}
$$

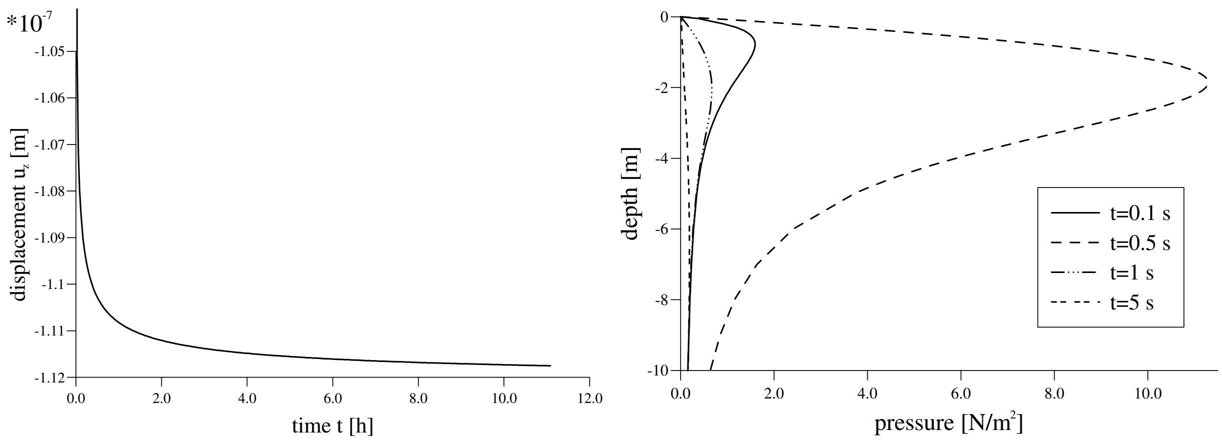
with the integration weights corresponding to (3), e.g.,

$$
\omega_n^{ef}\left(\hat{U}_{ij}^{s}, \mathbf{y}, \Delta t\right) = \frac{\mathscr{R}^{-n}}{L} \sum_{\ell=0}^{L-1} \int_{\Gamma_e} \hat{U}_{ij}^{s}\left(\mathbf{x}, \mathbf{y}, \frac{\gamma\left(\mathscr{R} e^{i\ell\frac{2\pi}{L}}\right)}{\Delta t}\right) N_e^f(\mathbf{x})\, \mathrm{d}\Gamma\, e^{-in\ell\frac{2\pi}{L}} . \tag{5}
$$

Note, the calculation of the integration weights is only based on the Laplace transformed fundamental solutions known from literature. After performing the numerical integrations a system of algebraic equations is obtained by the collocation method. Finally, a direct equation solver is applied.

CONSOLIDATION PROCESS IN A POROELASTIC HALF SPACE

In quasi-static poroelasticity, the main interest is in calculating the consolidation of the poroelastic material. To simulate this behavior, a poroelastic half space is loaded perpendicular to the surface by the total stress vector $t_z = -1000\,\mathrm{N/m^2} H(t)$, i.e., it is kept constant over time. The remaining surface is traction free and permeable, i.e., the pore pressure is assumed to be zero at the surface. The mesh is truncated outside an area of $6\,\mathrm{m} \times 15\,\mathrm{m}$ and the material data are those of soil ($K = 2.1 \cdot 10^8 \frac{\mathrm{N}}{\mathrm{m}^2}, G = 9.8 \cdot 10^7 \frac{\mathrm{N}}{\mathrm{m}^2}, \phi = 0.48, \alpha = 0.98, M = 5.24 \cdot 10^9 \frac{\mathrm{N}}{\mathrm{m}^2}, \kappa = 3.55 \cdot 10^{-9} \frac{\mathrm{m}^4}{\mathrm{Ns}}$).
In Fig. 1, the displacement at the surface in a point in $10\,\mathrm{m}$ distance to the load is depicted versus time. The consolidation process is obviously observed. Additionally to the settlement, the pore pressure distribution below the surface is given in Fig. 1 versus depth for four different times, i.e., at $t = 0.1\,\mathrm{s}, t = 0.5\,\mathrm{s}, t = 1\,\mathrm{s}$, and $t = 5\,\mathrm{s}$. As expected, the pore pressure



(a) Vertical surface displacement versus time

(b) Pressure below the surface versus depth

Figure 1. Numerical results for the half space

rises very fast from zero (the boundary condition at the surface) to its maximum value and tends afterwards to a constant value. For later times this maximum value is smaller and shifted to larger depth.
It should be remarked that the proposed algorithm is not sensitive to the time step size or shows any instabilities [4]. Further, not only for 3-d examples as above, also for 2-d the algorithm performs well.

References

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