# SHALLOW-WATER THEORY FOR WAVE-CURRENT-BOTTOM INTERACTIONS

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*Summary* A new shallow-water theory valid for wave-current-bottom interactions with arbitrary depth and unsteady horizontal currents is derived by Hamilton's canonical equations for surface waves, which constitutes a systematic hierarchy of partial differential equations for linear gravity waves in the near shore region. The first and second members of this hierarchy, the Helmholtz equation and the mild-slope equations of Berkhoff (1972) for pure waves and of Kirby (1984) with current, are second order. The third member is fourth order but may be approximated by Miles & Chamberlain's (1998) explicit fourth-order partial differential equation for pure waves which contains as a special case Chamberlain & Porter's (1995) modified mild-slope equation .

## INTRODUCTION

Wave-current-bottom interactions have all along received a widespread attention as main dynamical mechanism in coastal area. On the background of the mild-slope equation[1], Miles & Chamberlain[2] recently obtained a systematic hierarchy of partial differential equations for linear pure gravity waves in water of variable depth by using the expansion of the average Lagrangian, the resulting explicit forth-order partial differential equation is time-independent. Constructing the new structure of the unknown potential field, a more systematic hierarchy of time-dependent partial differential equation for wave-current-bottom interactions is developed by way of Hamilton's canonical equations[3], which effectively extends the system of Miles & Chamberlain[2].

## FORMULATION

We suppose that inviscid, incompressible fluid is in irrotational motion over a bed of varying depth $h(\mathbf{x})$, $\mathbf{x} \equiv (x, y)$ denoting horizontal Cartesian coordinates. The vertical coordinate, $z$ , is measured positively upwards with the free surface $z = \zeta(\mathbf{x}, t)$, $z = 0$ denoting the undisturbed free surface. Now a new determination of the structure of the unknown potential field $\Phi(\mathbf{x}, z, t)$ and $\zeta(\mathbf{x}, t)$ for wave-current-bottom interactions can be given as follows

$$\zeta = \zeta_0(\mathbf{x}, t) + \varepsilon\zeta_1(\mathbf{x}, t) \text{ , } \Phi = \phi_0(\mathbf{x}, t) + \varepsilon\left[\cosh \mathrm{k}(z - \zeta_0) + \kappa \mathrm{k}^{-1} \sinh \mathrm{k}(z - \zeta_0)\right]\phi_1(\mathbf{x}, t) \equiv \Re(\mathrm{k}^2, z)\phi_1 \quad (1)$$

where $\mathrm{k}^2 \equiv -\nabla^2 \equiv \left(-\partial^2/\partial x^2, -\partial^2/\partial y^2\right), \nabla \equiv (\partial/\partial x, \partial/\partial y)$, $\zeta_0$ and $\phi_0$ are the surface elevation due to presence of current and the velocity potential of the current, $\boldsymbol{U} = \nabla\phi_0$, $\varepsilon$ denotes the wave slope, $\kappa$ is determined by the relation

$$\kappa = k \tanh q = \omega_r^{\ 2}/g \qquad (q = k(h + \zeta_0)) \quad (2)$$

in which $k$ is the wavenumber and $\omega_r$ the relative frequency. The operators $\cosh \mathrm{k}(z - \zeta_0)$ and $\mathrm{k}^{-1} \sinh \mathrm{k}(z - \zeta_0)$ are defined by their power-series expansions in $\mathrm{k}^2$, and expand the operator $\Re$ in powers of the Helmholtz operator

$$\mathcal{H} \equiv \nabla^2 + k^2 = -\left(\mathrm{k}^2 - k^2\right) \quad (3)$$

Introducing the truncated expansion

$$\Phi(\mathbf{x}, z, t) = \left[\Re(k^2, z) - \left(\partial\Re/\partial \mathrm{k}^2\right)_{\mathrm{k}=k} \mathcal{H} + O(\mathcal{H}^2)\ \right]\phi_1(\mathbf{x}, t) \quad (4)$$

The classical Berkhoff mild-slope equation[1] for pure wave motion can be given as

$$\left(\nabla^2 + k^2\right)\psi = -A^{-1}\nabla A \cdot \nabla\psi \quad (5)$$

where $\Phi(\mathbf{x}, z, t) = \mathrm{Re}\left[f(h, z)\psi(\mathbf{x})\mathrm{e}^{-i\omega t}\right]$ with frequency $\omega$ , $A = (1/2k)\left[B + kh\left(1 - B^2\right)\right]$, $B = \tanh kh$, $f(h, z) = \cosh Q/\cosh kh$ , $Q = k(z + h)$. (5) suggests that

$$\mathcal{H}\phi_1 = -R^{-1}\nabla R \cdot \nabla\phi_1 \qquad \left(R = (1/2k)\left[T + q\left(1 - T^2\right)\right], T = \tanh q\right) \quad (6)$$

From (4) and (6), we obtain

$$\Phi(\mathbf{x}, z, t) = \phi_0 + \varepsilon\left[F(h, z)\phi_1 + G_1(h, z)\Psi_1 + G_2(h, z)\Psi_2 + G_3(h, z)\Psi_3 + O\left(|\nabla h|^2\right)\right] \quad (7)$$

where $F = \dfrac{\cosh Q}{\cosh q}$, $\Psi_1 = \nabla h \cdot \nabla\phi_1$ , $\Psi_2 = \nabla k \cdot \nabla\phi_1$ , $\Psi_3 = \nabla\zeta_0 \cdot \nabla\phi_1$ , $G_1 = \left(\dfrac{\partial\Re}{\partial \mathrm{k}^2}\right)_{\mathrm{k}=k} \dfrac{\partial R/\partial h}{R} = \dfrac{1}{2}\left[\dfrac{(Q - q)\sinh Q - \sinh q \sinh(Q - q)}{k^2 \cosh q}\right]\dfrac{\partial R/\partial h}{R}$ , $G_2 = \left(\dfrac{\partial\Re}{\partial \mathrm{k}^2}\right)_{\mathrm{k}=k} \dfrac{\partial R/\partial k}{R}$ , $G_3 = \left(\dfrac{\partial\Re}{\partial \mathrm{k}^2}\right)_{\mathrm{k}=k} \dfrac{\partial R/\partial \zeta_0}{R}$ .

Notice that $R = \int_{-h}^{\zeta_0} F^2 dz$. The total energy of the fluid $\mathrm{H}$ is written as

$$\mathrm{H} = (1/2)\rho \iint d\mathbf{x}\left\{g\zeta^2 + \int_{-h}^{\zeta} dz\left[(\nabla\Phi)^2 + \Phi_z^2\right]\right\} = \mathrm{H}_0 + \varepsilon\,\mathrm{H}_1 + \varepsilon^2\,\mathrm{H}_2 \qquad (\partial\Phi/\partial z \equiv \Phi_z) \quad (8)$$

From Hamilton's canonical equations for surface waves[3] ,we have

$$\rho\,\partial\zeta_1/\partial t = \delta\,\mathrm{H}_2/\delta\phi_1 \quad , \quad \rho\,\partial\phi_1/\partial t = -\delta\,\mathrm{H}_2/\delta\zeta_1 \qquad (9)$$

where $\delta$ denotes a variational derivative and $\rho$ fluid mass density.

## THE SHALLOW-WATER THEORY

Substituting (8) into (9) yields

$$\begin{aligned} \partial\zeta_1/\partial t &= -\zeta_1\left[k(\nabla\zeta_0\cdot\boldsymbol{U})\tanh q + \nabla\cdot\boldsymbol{U}\right] - \nabla\zeta_1\cdot\boldsymbol{U} + \int_{-h}^{\zeta_0} L dz - \nabla\cdot\int_{-h}^{\zeta_0} N dz + \delta P/\delta\phi_1 \\ \partial\phi_1/\partial t &= -g\zeta_1 - \nabla\phi_1\cdot\boldsymbol{U} + \phi_1 k(\nabla\zeta_0\cdot\boldsymbol{U})\tanh q \end{aligned} \qquad (10)$$

where the detailed expressions for $L$, $N$, and $P$ are given in Appendix. Elimination $\zeta_1$ from (10) leads to the time-dependent equation for the new shallow-water theory for wave-current-bottom interactions

$$\begin{aligned} &\frac{D^2\phi_1}{Dt^2} + (\nabla\cdot\boldsymbol{U})\frac{D\phi_1}{Dt} - \left\{\frac{D}{Dt}\left[k(\nabla\zeta_0\cdot\boldsymbol{U})\tanh q\right] + \left[k(\nabla\zeta_0\cdot\boldsymbol{U})\tanh q\right]\left[k(\nabla\zeta_0\cdot\boldsymbol{U})\tanh q + \nabla\cdot\boldsymbol{U}\right]\right\}\phi_1 \\ &\qquad + g\left[\int_{-h}^{\zeta_0} L dz - \nabla\cdot\int_{-h}^{\zeta_0} N dz + \frac{\delta P}{\delta\phi_1}\right] = 0 \qquad \left(\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{U}\cdot\nabla\right) \end{aligned} \qquad (11)$$

Accepting the common assumption for the mild-slope equation that terms with $\nabla F$, $\nabla h$, $\nabla k$, and $\nabla\zeta_0$ can be ignored, (11) reduced to the well-known Kirby mild-slope equation with current[4] which includes (5). When neglecting current $\boldsymbol{U}$ and $\zeta_0$, and considering purely harmonic motion, $\phi_1(\mathbf{x},t) = \mathrm{Re}\left[\Phi_0(\mathbf{x})e^{-i\omega t}\right]$, (11) leads to Mile & Chamberlain's explicit forth-order partial differential equation[2]

$$\begin{aligned} &(k^2 A - K)\Phi_0 + \nabla\cdot\left\{A\nabla\Phi_0 + \langle fG\rangle\nabla(\nabla h\cdot\nabla\Phi_0) + \left[M(\nabla h\cdot\nabla\Phi_0) - \right.\right. \\ &\qquad\qquad \left.\left.\nabla\cdot\left(\langle G^2\rangle\nabla(\nabla h\cdot\nabla\Phi_0) + \langle fG\rangle\nabla\Phi_0\right)\right]\nabla h\right\} = 0 \end{aligned} \qquad (12)$$

($A \equiv H$ and $f = F$ in Mile & Chamberlain's notation). The detailed expressions for $K$, $M$, $G$ and $\langle(\ )\rangle$ are given in [2]. Discarding all terms of $G$ reduces (12) to Chamberlain & Porter's modified mild-slope equation[5].

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## APPENDIX: EXPRESSIONS FOR $L$, $N$ AND $P$ IN (10) AND (11)

$$L = \phi_1(\nabla F)^2 + \nabla F\cdot(F\nabla\phi_1 + \Psi_1\nabla G_1 + \Psi_2\nabla G_2 + \Psi_3\nabla G_3) + \phi_1 F_z^2 + F_z(\Psi_1 G_{1z} + \Psi_2 G_{2z} + \Psi_3 G_{3z}),$$

$$\begin{aligned} N &= F^2\nabla\phi_1 + F(\phi_1\nabla F + \Psi_1\nabla G_1 + \Psi_2\nabla G_2 + \Psi_3\nabla G_3) + \nabla(F\phi_1)\cdot(\nabla h\nabla G_1 + \nabla k\nabla G_2 + \nabla\zeta_0\nabla G_3) + \\ &\quad \left\{\Psi_1\left[(\nabla G_1)^2 + G_{1z}^2\right] + \Psi_2 G_{1z}G_{2z} + \Psi_3 G_{1z}G_{3z}\right\}\nabla h + \left\{\Psi_2\left[(\nabla G_2)^2 + G_{2z}^2\right] + \Psi_1 G_{1z}G_{2z} + \right. \\ &\quad \left.\Psi_3 G_{2z}G_{3z}\right\}\nabla k + \left\{\Psi_3\left[(\nabla G_3)^2 + G_{3z}^2\right] + \Psi_1 G_{1z}G_{3z} + \Psi_2 G_{2z}G_{3z}\right\}\nabla\zeta_0 + \nabla h\nabla G_1\cdot(\Psi_2\nabla G_2 + \Psi_3\nabla G_3) + \\ &\quad \nabla k\nabla G_2\cdot(\Psi_1\nabla G_1 + \Psi_3\nabla G_3) + \nabla\zeta_0\nabla G_3\cdot(\Psi_1\nabla G_1 + \Psi_2\nabla G_2)\ , \end{aligned}$$

$$\begin{aligned} P &= \iint d\boldsymbol{x}\int_{-h}^{\zeta_0} dz\left\{(1/2)\left[G_1^2(\nabla\Psi_1)^2 + G_2^2(\nabla\Psi_2)^2 + G_3^2(\nabla\Psi_3)^2\right] + G_1G_2\nabla\Psi_1\cdot\nabla\Psi_2 + G_1G_3\nabla\Psi_1\cdot\nabla\Psi_3 + \right. \\ &\quad \left. G_2G_3\nabla\Psi_2\cdot\nabla\Psi_3 + (G_1\nabla\Psi_1 + G_2\nabla\Psi_2 + G_3\nabla\Psi_3)\cdot\left[\nabla(\phi_1 F) + \Psi_1\nabla G_1 + \Psi_2\nabla G_2 + \Psi_3\nabla G_3\right]\right\}. \end{aligned}$$

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