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UNSTEADY UNDULAR BORE TRANSITION IN FULLY NONLINEAR DISPERSIVE WAVE DYNAMICS

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Summary A complete set of conditions describing the expanding undular bore transition in nonlinear conservative wave dynamics is derived. The transition conditions are obtained in a general form by employing the asymptotic formulation of the problem based on the Whitham modulation equations and allow, in particular, determination of the lead solitary wave amplitude as a function of the initial jump in (generally) non-integrable systems. Several examples pertaining to the shallow water dynamics and collisionless plasma physics are considered.

BACKGROUND

Undular bores (in diffrent contexts also called dispersive, or collisionless shocks) represent regions of rapidly oscillating nonlinear wave structures evolving out of smooth initial or boundary conditions of nonlinear dispersive wave equations. In the presence of small dissipation, the undular bores despite their oscillatory structure exhibit global properties characteristic for classical, turbulent bores or shock waves: they have steady, though wavy, profile and constant width such that the speed of the bore propagation and the transition conditions can be derived within the classical theory of hyperbolic conservation laws. The qualitative theory of such undular bores has been first developed by Benjamin and Lighthill [1] for shallow water waves and by Sagdeev [2] for rarefied plasma flows.

This classical theory of steady undular bores, however, is valid only for the regime, when nonlinearity, dispersion and dissipation are in balance. Contrastingly, in the case when dispersive effects dominate dissipation, the traditional analysis of the mass, the momentum and the energy balance across the undular bore transition can not be applied directly. The reason for that is that *the boundaries of the dissipationless undular bore diverge with time*, i.e instead of the single shock speed U defined by the mass balance one has now two different speeds $s_1 > s_2$ determining motion of the undular bore boundaries. These speeds, however, can not be obtained without the analysis of the nonlinear oscillatory structure of the undular bore because in the conservative dynamics, in contrast to the steady, dispersion-dissipative case, dispersion not only dramatically modifies the fine structure of the flow in the bore region but also, along with nonlinearity, determines the undular bore location.

In a weakly nonlinear case, when the original system can be approximated by one of the exactly integrable equations, such as Korteweg – de Vries or nonlinear Schrödinger equation, the undular bore solutions can be found using semi-classical asymptotic of the inverse scatering transform method [3]. Fully nonlinear dispersive waves, however, are often described by non-integrable systems, for which exact solutions of the Cauchy problem are not available in principle. In this case determination of the transition conditions allows one to put the entire problem of the dissipationless undular bore dynamics into the classical hydrodynamic setting, so that the expanding undular bore is 'embedded' into the solution of the dispersionless inviscid Euler equations bypassing the analysis of its fine structure.

GENERAL SETTING AND FORMULATION OF THE PROBLEM

We consider the system describing fully nonlinear flows in a dissipationless dispersive medium. In a symbolic form such a system can be represented as

$$\mathbf{K}_{M,N}(\mathbf{U};\partial_t\mathbf{U},\partial^2_{tt}\mathbf{U},\ldots;\partial_x\mathbf{U},\partial^2_{xx}\mathbf{U},\ldots)=0\,, \tag{1}$$

where $\mathbf{U}$ and $\mathbf{K}$ are vectors and M and N are the orders of the system with respect to the t- and x- derivatives respectively. In this study, we restrict ourselves with the important subclass of such systems with $M=2$, $N=4$ and the real-valued linear dispersion relation $\omega=\omega_0(k)$, where ω, is the frequency and k is the wavenumber. We assume that the system (1) supports nonlinear single-phase travelling wave solutions $f(kx-\omega t)$ and possesses at least four independent conservation laws of the form $\partial_t P_j+\partial_x Q_j=0$.

We define the dispersionless limit of the system (1) by introducing $X=\epsilon x$ and $T=\epsilon t$ instead of x and t, and then by tending $\epsilon\to 0$. Let this limit have the form of the Euler equations of ideal gas dynamics

$$\partial_T\rho+\partial_X(\rho u)=0\,,\qquad \partial_T u+u\partial_X u+c_s^2(\rho)\rho^{-1}\partial_X\rho=0, \tag{2}$$

where ρ is the 'flow density', u is the 'flow velocity', and $c_s(\rho)$ is the 'sound speed'.

The described class of systems is quite broad and includes some known integrable models such as defocusing nonlinear Schrödinger equation and Kaup-Boussinesq system. As a typical (and important) non-integrable example one can indicate the Green-Naghdi system for fully nonlinear shallow water gravity waves [4] which also appears in such different physical contexts as bubbly fluid dynamics and magnetohydrodynamics of solar tachocline:

$$\partial_t h+\partial_x(hu)=0\,,\qquad \partial_t u+u\partial_x u+\partial_x h=(3h)^{-1}\partial_x[h^3(u_{xt}+uu_{xx}-{u_x}^2)]\,. \tag{3}$$

Here h is total liquid depth and u is horisontal velocity averaged over depth (in the traditional shallow water terms).
The requirement of the particular form (2) for the dispersionless limit is, actually, not necessary for our theory to be valid, it is chosen solely for a greater transparency of presentation.
The problem formulation: *to derive a set of conditions determining transition for the solutions of the system (1) from one constant state* $(\rho, u) \to (\rho_2, u_2)$ *as* $x \to -\infty$ *to another constant state* $(\rho, u) \to (\rho_1, u_1)$ *as* $x \to +\infty$ provided $\rho_2 > \rho_1$.

RESULTS

The sought set of conditions is found by adopting the approach to the description of undular bores established in the integrable systems theory. The main assumptions we make are the following: (i) asymptotic as $t \to \infty$ description of the undular bore with the aid of the modulated travelling wave solutions of (1) $f(kx - \omega t; X, T)$ where $X = \epsilon x$ and $T = \epsilon t$, $\epsilon \ll 1$; (ii) hyperbolicity of the corresponding modulation (Whitham) equations $\partial_T \bar{P}_j + \partial_X \bar{Q}_j = 0$, $j = 1, \dots, 4$, where the bar denotes averaging over the family (1) [6].
We introduce the "extended" linear dispersion relation $\omega = \omega_0(k, \bar{\rho}, \bar{u})$ for the system (1) by linearizing it about the slowly changing mean state $\rho = \bar{\rho}(X, T)$, $u = \bar{u}(X, T)$. Then, analysis of the natural matching of the characteristics for the solutions of the Whitham and Euler equations at the (unknown at the onset) boundaries of the undular bore leads to the following description of the undular bore transition. The undular bore asymptotically as $t \to \infty$ occupies the self-similarly expanding zone $[s_2, s_1]$, where $s = X/T$, such that the following conditions hold.
(i) Relationship between admissible values of the boundary parameters $\rho_{1,2}$, $u_{1,2}$:

$$u_2 - u_1 = \int_{\rho_1}^{\rho_2} \frac{c_s(\rho)}{\rho} d\rho \,. \qquad (4)$$

(ii) Location of the undular bore edges $s_{1,2}$ in self-similar co-ordinate $s = X/T$ is determined from:

$$s_2 = \frac{\partial \Omega_0}{\partial k}(k_2, \rho_2)\,, \qquad s_1 = \frac{\tilde{\Omega}_s(k_1, \rho_1)}{k_1}\,, \qquad (5)$$

where

$$\Omega_0(k, \bar{\rho}) = \omega_0(k, \bar{\rho}, \bar{u}(\bar{\rho})) \qquad \tilde{\Omega}_s(\tilde{k}, \bar{\rho}) = -i\Omega_0(i\tilde{k}, \bar{\rho})\,, \qquad (6)$$

$$\bar{u}(\bar{\rho}) = u_1 + \int_{\rho_1}^{\bar{\rho}} \frac{c_s(\rho)}{\rho} d\rho\,, \qquad k_2 = k(\rho_2)\,, \;\; \tilde{k}_1 = \tilde{k}(\rho_1)\,. \qquad (7)$$

The functions $k(\bar{\rho})$ and $\tilde{k}(\bar{\rho})$ are found from the ordinary diffrential equations:

$$\frac{dk}{d\bar{\rho}} = \frac{\partial \Omega_0 / \partial \bar{\rho}}{\bar{u}(\bar{\rho}) + c_s(\bar{\rho}) - \partial \Omega_0 / \partial k}\,, \qquad k(\rho_1) = 0\,. \qquad (8)$$

$$\frac{d\tilde{k}}{d\bar{\rho}} = \frac{\partial \tilde{\Omega}_s / \partial \bar{\rho}}{\bar{u}(\bar{\rho}) + c_s(\bar{\rho}) - \partial \tilde{\Omega}_s / \partial \tilde{k}}\,, \qquad \tilde{k}(\rho_2) = 0\,. \qquad (9)$$

(iii) Inequalities providing consistency of (i) and (ii) should be satisfied:

$$u_2 - c_s(\rho_2) < s_2 < u_2 + c_s(\rho_2)\,, \qquad s_1 > u_1 + c_s(\rho_1)\,. \qquad (10)$$

Remarkably, the conditions (i) – (iii) are determined by the dispersonless limit (2), which is characterised by the 'sound speed' $c_s(\rho)$, and the linear dispersion relation $\omega_0(k; \bar{\rho}, \bar{u})$ and thus, do not rely on the integrability of the system (1). We also note that the condition (i) has been for the first time proposed [7]in the context of the dissipationless dispersive shocks in plasma.
We apply conditions (i) – (iii) to the systems where undular bore solutions have been obtained earlier using either exact analytic methods (the integrable Kaup-Boussinesq system for the dispersive shallow water waves [5]) or numerically (fully nonlinear ion-acoustic waves in collisionless plasma [7]). In both cases a complete agreement of our conditions with the earlier results is shown. We then apply the undular bore conditions (i)– (iii) to the Green – Naghdi system (3) to obtain the lead solitary wave amplitude in the fully nonlinear shallow water undular bore as a function of the initial jump.

References

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