EXACT TRACKING CONTROL FOR NONLINEAR STRUCTURAL AND MECHANICAL SYSTEMS

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Summary A new methodology for the *exact* tracking control of nonlinear structural and mechanical systems with holonomic and/or nonholonomic trajectory requirements is presented.

We consider in this paper an n degree-of-freedom structural or mechanical system described by Lagrange's equations [1] as

$$M(x,t)\ddot{x} = F(x,\dot{x},t)\,, \quad x(0) = x_0,\ \ \dot{x}(0) = \dot{x}_0\,. \tag{1}$$

Here, x is the *generalized* coordinate used in describing the configuration of the system. The n by n matrix M is positive definite and, in general, is a function of x and the time, t. We shall assume that the observation m-vector, $y(t)$, is related to the response $x(t)$ of the system by the observation equation

$$y(t) = Cx(t) \tag{2}$$

where C is the constant m by n measurement matrix. It is desired to control the system described by (1), and determine the control force vector $F^c(y,\dot{y},t)$ so that for the controlled system described by,

$$M(x,t)\ddot{x} = F(x,\dot{x},t) + F^c(y,\dot{y},t)\,, \tag{3}$$

the measurement $y(t)$ satisfies the s desired tracking relations

$$h(y,\dot{y},t) = 0\,, \tag{4}$$

where h is an s-vector. We assume that the functions $h_i(y,\dot{y},t)$ are C^1 and that (4) constitute a set of relations that are feasible for the system to satisfy. The set (4) may contain relations that are integrable and/or non-integrable. Also, the system's initial conditions $x(0)$ and $\dot{x}(0)$ are such as to satisfy equation (4). One can relax this assumption on the initial conditions [2]. Differentiating equation (4) with respect to time, one obtains

$$H(y,\dot{y},t)\ddot{y} = \frac{\partial h}{\partial \dot{y}}\ddot{y}(t) = -\frac{\partial h}{\partial y}\dot{y} - \frac{\partial h}{\partial t} \tag{5}$$

where we have denoted the s by m matrix $\frac{\partial h}{\partial \dot{y}}$ by H. In view of (2) this can be expressed as

$$B\ddot{x} = HC\ddot{x}(t) = b(y,\dot{y},t) = b(Cx, C\dot{x}, t)\,, \tag{6}$$

where we have denoted the s by n matrix HC by B, and the s-vector b is given by

$$b(y,\dot{y},t) = -\frac{\partial h}{\partial y}\dot{y} - \frac{\partial h}{\partial t}\,. \tag{7}$$

We note that by using relations (2) and (4) on the right hand side of (7), the s-vector b may be thought of as a known function of x, $\dot{x}$ and t, as explicitly indicated in (6). From here on, we shall suppress the arguments of the various quantities unless required for clarification. We now state the following central result [2].

<u>**Main Result**</u> The class of control forces that minimize *at each instant of time* the quantity

$$J(t) = (F^c)^T N(x,t) F^c\,, \tag{8}$$

where the n by n matrix $N(x,t)$ is a given positive definite matrix at each instant of time t, while causing the controlled system (3) to satisfy relation (6), is given by

$$F^c = N^{-1/2} G^{+} (b - BM^{-1}F) \tag{9}$$

where, G denotes the matrix $B(N^{1/2}M)^{-1}$, and G^{+} denotes the Moore-Penrose generalized inverse of the matrix G.

<u>**Proof**</u>: For brevity, the proof will not be given here; it will be presented at the meeting. ❑

Example Consider a nonlinear mechanical system whose equation of motion is given by ($M = I$)

$$\ddot{x} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} s(x_2 - x_1) - c_1\dot{x} \\ lx_1 - x_1x_3 - x_2 - c_2\dot{x}_2 \\ F_3 = x_1x_2 - bx_3 - c_3\dot{x}_3 \end{bmatrix} = F, \; x(0) = x_0, \dot{x}(0) = \dot{x}_0. \tag{10}$$

We shall assume that the measurement m-vector, y, is given by the relation

$$y = Cx = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} x, \tag{11}$$

and the desired (and feasible), nonholonomic trajectory we want the system (10) to track is given to be

$$\dot{y}_2 = y_3\dot{y}_1. \tag{12}$$

We wish to apply a control force, F^c, so that in the presence of F, relation (12) is 'exactly' satisfied. For illustration, we take: $s = l = 1$, and $b = 2$; $c_1 = c_3 = 1/2$, and $c_2 = 0$; and, $x_1(0) = 0$, $x_2(0) = 1$, $x_3(0) = 1$, $\dot{x}_1(0) = \dot{x}_2(0) = \dot{x}_3(0) = 0$. Using numerical integration, Figure 1(a) shows the error in the satisfaction of this trajectory requirement, (12), without any control. Using the control force, F^c, that is explicitly given by equation (9), we see, from Figure 1(b), that this tracking error is reduced to within the numerical precision (10^{-8}) with which the equations of motion are integrated.

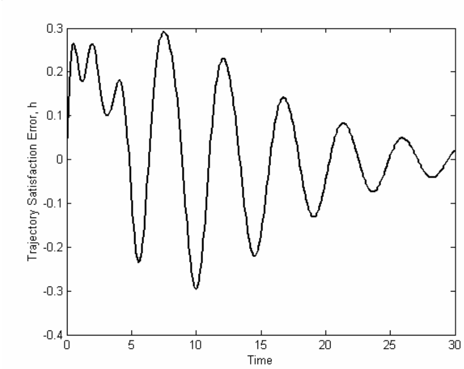

Figure 1(a)

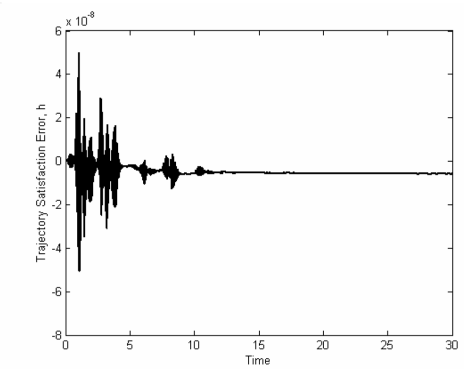

Figure 1(b)

Conclusions

This paper provides a powerful, new methodology for controlling nonlinear structural and mechanical systems. It has been inspired by recent results in analytical dynamics. The methodology has the following advantages: (1) the control force required to control the *nonlinear* system so that it satisfies the given nonholonomic and/or holonomic trajectory requirements is obtained *explicitly* in closed-form; (2) this control force, theoretically speaking, will cause the trajectory requirements to be 'exactly' satisfied by the nonlinear controlled system; (3) it allows the weighted norm of the control force to be minimized *at each instant of time*--most control methods minimize time integrals of such norms; (4) the computations required to determine the control force involve simple matrix multiplications and additions, making the method attractive for real-time, on-line control of nonlinear mechanical systems. An example of the control methodology is illustrated. It has also been successfully used for precision motion-control of multi-arm robots, for on-orbit formation-flight of satellites, and for controlling flexible, nonlinear vibrating systems.

References

[1] Udwadia F. E., Kalaba R. E.: What is the General Form of the Equation of Motion for Constrained Mechanical Systems? *Journal of Applied Mechanics* **69**:335-339, 2002.
[2] Udwadia F. E.: A New perspective on the Tracking Control of Nonlinear Structural and Mechanical Systems. *Proceedings of the Royal Society of London, Series A* **459**:1783-1800, 2003.