GAUGE PRINCIPLE FOR IDEAL FLUIDS AND VARIATIONAL PRINCIPLE

Tsutomu Kambe[1]
IDS, Higashi-yama 2-11-3, Meguro-ku, Tokyo 153-0043, Japan

Summary A variational formulation is given for flows of a compressible ideal fluid by defining a Galilei-invariant Lagrangian. Variations are required to be gauge-invariant with respect to both translation and rotation groups. Carrying out variations by using a covariant derivative defined in terms of gauge fields, we deduce the Euler's equation of motion. Noether's theorem results in the conservation laws of momentum and angular momentum.

INTRODUCTION: FLUID FLOWS AND FIELD THEORY

Study of fluid flows is considered to be a field theory in Newtonian mechanics. In other words, it is a *field theory of mass flow* subject to Galilei transformation. It is well-known that there are various similarities between fluid mechanics and electromagnetism. For example, the functional relation between velocity and vorticity fields is the same as the Biot-Savart law known in the electromagnetism between magnetic field and electric current. One may ask whether the similarity is mere an analogy, or has a solid theoretical background.

In the theory of *gauge field*, a guiding principle is that laws of physics should be expressed in a form that is independent of any particular coordinate system. In the quantum field theory, a free-particle Lagrangian is defined first in such a way as having an invariance under Lorenz transformation. Next, a gauge principle is applied to the Lagrangian, requiring it to have a *symmetry*, *i.e.* the gauge invariance. As a result, a gauge field such as the elecromagnetic field is introduced to satisfy *local* gauge invariance. In regard to the fluid flows, relevant symmetry groups are *translation group* and *rotation group* [3].

We seek a scenario which has a formal equivalence with the gauge theory in the quantum field theory. To that end, we define a Galilei-invariant Lagrangian for fluid flows which has also global gauge invariance. The *global* means that transformations are uniform at all points and times. Next, we examine whether it has *local* gauge invariance. Applying the gauge principle to the Lagrangian first with respect to translational transformations, the action principle results in the equation of motion for *irrotational flows.* That is, the velocity field thus obtained from the translation invariance must have a potential. This corresponds to *superfluid* flows.

Next, we consider an additional transformation with respect to the gauge group $SO(3)$, a *rotation group* in the three-dimensional Euclidean space. The gauge transformation introduces a new *rotational* component in the velocity field (*i.e.* the vorticity), even though the original field is irrotational. In complying with the local gauge invariance, a gauge-covariant derivative is defined by introducing a new gauge field Ω. Galilei invariance of the covariant derivative requires that the *gauge field* Ω should coincide with the *vorticity* [1, 2]. As a result, the covariant derivative of velocity is found to be the so-called *material* derivative of velocity, and the Euler's equation of motion for an ideal fluid is derived from the Hamilton's principle. The Noether's theorem leads to conservation laws associated with gauge invariances: *i.e.* conservation equations of momentum and angular momentum. Furthermore, the Lagrangian has a symmetry with respect to particle permutation, which leads to a local law of vorticity conservation, *i.e.* the vorticity equation [2, 3].

The flow fields are characterized by two gauge groups: a translation group and a rotation group. The former is abelian and the latter is non-abelian. Thus, the flow fields are governed by two characteristically different transformation laws.

HAMILTON'S PRINCIPLE FOR AN IDEAL FLUID

Constitutive conditions and definitions

We carry out the material variations under the following conditions and definitions.
(*i*) *Kinematic condition*: The $\boldsymbol{x}$-space trajectory of a material particle, specified by the Lagrangian coordinte $\boldsymbol{a}$, is denoted by $\boldsymbol{x}_a(\tau) = \boldsymbol{x}(\tau, \boldsymbol{a})$ and the time $t = \tau$, and the particle velocity is

$$\boldsymbol{v}(\boldsymbol{x}, t) = \partial_\tau \boldsymbol{x}(\tau, \boldsymbol{a})\,. \tag{1}$$

All the variations are taken so as to follow such trajectories of material particles. In addition, all the analyses are carried out by keeping mass fixed. As a consequence, the equation of continuity must be satisfied always.
(*ii*) *Gauge-covariance*: All the expressions must satisfy both global and local gauge invariance.
(*iii*) *Hamilton's principle*: Lagrangian for flows of an ideal fluid is defined by

$$L_{\mathrm{F}} := \int_M \tfrac{1}{2}\,\langle\, \boldsymbol{v}, \boldsymbol{v}\,\rangle\, \rho \,\mathrm{d}V - \int_M \epsilon(\rho, s)\, \rho\, \mathrm{d}V\,, \qquad \langle \boldsymbol{v},\, \boldsymbol{v}\rangle = \sum v_k v_k, \tag{2}$$

where $\boldsymbol{v} = (v_k)$ is the fluid velocity, ρ the density, and ϵ the internal energy per unit fluid mass, with $\mathrm{d}V$ a volume element, and M is a bounded space under consideration with $\boldsymbol{x} \in M \subset \mathbb{R}^3$. The action principle is given by $\delta\mathcal{A} = 0$, where the action is defined by $\mathcal{A} = \int_{t_0}^{t_1} L_{\mathrm{F}}[\boldsymbol{v}, \rho, \epsilon]\ \mathrm{d}t$.

[1]kambe@gate01.com Visiting Professor, Nankai Institute of Mathematics (China)

(*iv*) *Covariant derivative*: Gauge-covariant derivative $\nabla_t \boldsymbol{v}$ must be used for the variation, where

$$\nabla_t \boldsymbol{v} = \partial_t \boldsymbol{v} + \mathrm{grad}(\tfrac{1}{2}\, v^2) + \boldsymbol{\omega} \times \boldsymbol{v} = \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v}\,, \qquad (\boldsymbol{\omega} = \mathrm{curl}\, \boldsymbol{v}). \tag{3}$$

(*v*) *Physical material*: An ideal fluid is defined by the property that there is no disspative mechanism within it such as viscous dissipation or thermal conduction. As a consequence, the fluid motion is *isentropic*, *i.e.* the entropy s per unit mass remains constant following the motion of each material particle. The entropy is not necessarily constant at every material particle, *i.e.* not necessarily homentropic.

We carry out an *isentropic material variation* satisfying local gauge invariance, under the above conditions.

Gauge principle

For the gauge transformations, it is important to recognize that the velocity $\boldsymbol{v}$ in the Lagrangian (2) must be represented by the form (1). According to the gauge principle, the partial time derivative $\partial/\partial t$ must be modified so that all the variations as well as the Lagrangian L_{F} are gauge-invariant. The above form of the covariant derivative $\nabla_t \boldsymbol{v}$ can be deduced on this basis of the gauge principle.

Suppose that the time derivative of velocity $\boldsymbol{v}$ is represented such that

$$\nabla_t \boldsymbol{v} = \mathrm{D}_t \boldsymbol{v} + \Omega \boldsymbol{v} = \partial_t \boldsymbol{v} + A \boldsymbol{v} + \Omega \boldsymbol{v}, \qquad \mathrm{D}_t \boldsymbol{v}_* = \partial_t \boldsymbol{v}_* + A \boldsymbol{v}_*.$$

where $\boldsymbol{v}_* = \mathrm{grad} f$, and A and Ω are linear operators called *gauge fields*, and the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ can be represented with a linear combination of irrotational and rotational parts: $\boldsymbol{v}(\boldsymbol{x}, t) = \mathrm{grad} f + \mathrm{curl}\, \boldsymbol{B}$ for $f(\boldsymbol{x})$, $\boldsymbol{B}(\boldsymbol{x}) \in C^\infty[M]$. First, concerning an irrotational flow field $\boldsymbol{v}_* = \mathrm{grad} f$, we require that the covariant derivative $\mathrm{D}_t \boldsymbol{v}_* = \partial_t \boldsymbol{v}_* + A \boldsymbol{v}_*$ is invariant with respect to *translational* gauge transformation, both global and local. Furthermore, $\mathrm{D}_t \boldsymbol{v}_*$ is required to be invariant with respect to Galilei transformation. This determines the gauge operator A such that $A \boldsymbol{v}_* = (\boldsymbol{v}_* \cdot \nabla) \boldsymbol{v}_* = \frac{1}{2}\, \mathrm{grad}(\boldsymbol{v}_*)^2$, where $(\boldsymbol{v}_* \cdot \nabla)(\boldsymbol{v}_*)_k = (\partial_i f) \partial_i (\partial_k f) = (\partial_i f) \partial_k (\partial_i f) = \frac{1}{2}\, \partial_k (\boldsymbol{v}_*)^2$.

Next, it is required that the covariant derivative $\nabla_t \boldsymbol{v} = \mathrm{D}_t \boldsymbol{v} + \Omega \boldsymbol{v}$ is invariant with respect to *rotational* transformation $SO(3)$, both global and local. Furthermore, requirement of Galilei invariance of $\mathrm{D}_t \boldsymbol{v} + \Omega \boldsymbol{v}$ determines the gauge operator Ω which is represented as a skew-symmetric matrix (an element of Lie algebra $\mathbf{so}(3)$). Then, the $\Omega \boldsymbol{v}$ is expressed by a vector product $\hat{\Omega} \times \boldsymbol{v}$, where

$$\hat{\Omega} = \nabla \times \boldsymbol{v} = \boldsymbol{\omega}$$

is the **vorticity**. It is found that the vorticity $\boldsymbol{\omega}$ is the gauge field with espect to the gauge group $SO(3)$. Thus, the covariant derivative (3) is deduced. It should be noted that the covariant derivative $\nabla_t \boldsymbol{v}$, *i.e.* the *Lagrange derivative* of velocity vector, is invariant with respect to the two gauge transformations: both translational and rotational. Thus it is verified that the Lagrange derivative of velocity has a *gauge-theoretic* significance.

Variation of the action $\mathcal{A}$ is given by

$$\delta\, \mathcal{A} = \Big[\int_M \langle \boldsymbol{v}, \boldsymbol{\xi} \rangle\, \rho\, \mathrm{d}V \Big]_{t_0}^{t_1} + \int_{t_0}^{t_1} \mathrm{d}t \oint_S p\, \langle \boldsymbol{n}, \boldsymbol{\xi} \rangle\, \mathrm{d}S - \int_{t_0}^{t_1} \mathrm{d}t \int_M \big\langle \big(\nabla_t \boldsymbol{v} + \rho^{-1} \mathrm{grad}\, p \big), \, \boldsymbol{\xi}\, \big\rangle\, \rho\, \mathrm{d}V,$$

where p is the pressure. The first two terms vanish owing to the boundary conditions of *vanishing* of the variation $\boldsymbol{\xi}$. Thus, the action principle $\delta\, \mathcal{A} = 0$ for arbitrary variation $\boldsymbol{\xi}$ leads to

$$\nabla_t \boldsymbol{v} + \rho^{-1}\, \nabla\, p = 0\,, \tag{4}$$

This is the *Euler's equation of motion*. Using (3), we have $\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = -\rho^{-1}\, \nabla\, p$, or equivalently $\partial_t \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{v} + \nabla(\frac{1}{2}\; v^2) = -\nabla\, h$, where $(1/\rho) \nabla p = \nabla h$ (h: enthalpy). The Euler's equation (4) must be supplemented by the equation of continuity and the isentropic equation:

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{v}) = 0, \qquad \mathrm{D}_t s = \partial_t s + \boldsymbol{v} \cdot \nabla\, s = 0\,.$$

CONCLUSION

Thus, guided by the gauge principle in the quantum field theory, one can carry out a gauge-covariant variational formulation of ideal fluid flows consistently.

References

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